Financial Mathematics.
TD 1 & 2, 2024
Introduction to financial derivatives, one-period model.

EXERCISE 1 -Future on a dividend-paying asset

1. Prove that the forward price (term T) of a tradeable asset is

$$K = e^{rT}(S_0 - I),$$

where S_0 is the spot price, I is the value at $t = 0$ of the dividends (known coupons) paid by the underlying asset between 0 and T (in this case $I > 0$) or the storage cost between 0 and T, which is paid at $t = 0$ (in this case $I < 0$).

2. A treasury bond with a nominal value of EUR 100 and a nominal rate of 8% pays its next coupon in 3 months. Calculate the forward price of this bond with a 6-month maturity if its price today is 106 EUR and the interest rate is 5% per annum.

EXERCISE 2 -Put-Call Parity.
We consider a risky asset whose price at time t is S_t. We assume that the interest rate r is positive. We note $c(t, S_t, T, K)$ (respectively $p(t, S_t, T, K)$) the price of a European *call* (respectively of a *put* European) with strike price K, maturity T and whose underlying asset is S.

1. Show the Put-Call parity property of European options prices :

$$c(0, S_0, T, K) - p(0, S_0, T, K) = S_0 - Ke^{-rT}.$$

2. Deduce that the price of the *call* satisfies

$$(S_0 - Ke^{-rT})^+ \leq c(0, S_0, T, K) \leq S_0.$$

Show directly (without using 1.) the inequality $S_0 - Ke^{-rT} \leq c(0, S_0, T, K)$.

3. We assume that the risky asset trades at 20 euros, that the price of a European *call* on this asset, with strike price $K = 11$ euros and maturity $T = 1$ year , is 13 euros. We further assume that $r = 9.531\%$. Compute the price of a European *put* with the same characteristics.

4. Show that today's prices $C_0(T, K) = C(0, S_0, T, K)$ and $P_0(T, K) = P(0, S_0, T, K)$ of the *American call* and *American put options* with maturity T and strike price K satisfy

$$C_0(T, K) - P_0(T, K) \leq S_0 - Ke^{-rT}.$$

5. In the Black Scholes model, $S_t = S_0 e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}$, where $(W_t)_{t\geq 0}$ is a Brownian motion. Knowing that W_t is a centered normal random variable with variance t, calculate $\mathbb{E}(e^{-rT}(S_T - K)^+) - \mathbb{E}(e^{-rT}(K - S_T)^+)$. Conclude that the *insurance* approach - which mainly consists in calculating the price as the discounted expectation of the future *payoff* - leads to a contradiction with the European Call-Put parity if $\mu \neq r$.

EXERCISE 3 - Using the same notations as in Exercise 2, show that

1. For any maturity $T > 0$, the price of *call* and *put* are convex at the strike price K.
2. $\forall K > 0$, $\forall 0 \leq T_1 \leq T_2$, $c(0, S_0, T_1, K) \leq c(0, S_0, T_2, Ke^{r(T_2 - T_1)})$.

EXERCISE 4 - Butterfly options.

Given an asset whose price at time T is S_T and strikes $K_1 < K_2 < K_3$, a *butterfly* is a combination of *trading* which is the result of the following net position : a long position on a European *call* with strike price K_1, a long position on a European *call* with strike price K_3, and a position short on two European *calls* with strike price K_2.

1. What is the *payoff* of such an option ? Calculate its price for all $t \leq T$.
2. If K_2 is the middle of the interval $[K_1, K_3]$, show that the *butterfly* can be created by buying and selling options *put* with the different prices of exercise K_1, K_2 and K_3.

NB : *butterfly* options can be bought when the investor believes that the underlying asset will not rise or fall much.

EXERCISE 5 -Effects of dividends on the price of European options.
In this exercise we consider options whose underlying asset pays dividends. Let D be the present value of all the dividends paid over the interval $[0, T]$. Show the relationships :

1. $c_0(T, K) \geq S_0 - D - Ke^{-rT}$,
2. $p_0(T, K) \geq D + Ke^{-rT} - S_0$,
3. (modified Put-Call parity) $c_0(T, K) + D + Ke^{-rT} = p_0(T, K) + S_0$.
4. If the risky asset pays a dividend at time t, show that the price of the *call* remains continuous at t, even if the price of the risky asset is not continuous at t.

EXERCISE 6 - A given asset trades at 95 euros and the European *calls* and *puts* on the given asset, with a strike price of 100 and a maturity of three months, trade respectively at 1.97 euros and 6.57 euros. In one month, the asset will pay a dividend of 1 euro. The prices of the one-month and three-month zero-coupon bonds are 99.60 and 98.60 respectively. Build an arbitrage strategy, if possible.

EXERCISE 7 -American Options We denote by $\text{Call}_t(T, K)$ the price at date t of an European call with strike K and expiry date T, and by $\text{Put}_t(T, K)$ the one of a put with same maturity T and strike K. $\text{CallAmer}_t(T, K)$ and $\text{PutAmer}_t(T, K)$ correspond respectively to the American call and put. We assume that the underlying asset does not pay dividends.

1. Show that for all $t \leq T$,
$$\text{CallAmer}_t(T, K) = \text{Call}_t(T, K).$$
2. Show that for all $t \leq T$,
$$\text{Put}_t(T, K) \leq \text{Amer Put}_t(T, K) \leq \text{Put}_t(T, K) + K(1 - e^{-r(T-t)}).$$

EXERCISE 8 -A one-period model
We consider a one-period market with three states $\omega_1, \omega_2, \omega_3$ and two risky assets :

— An asset S with a value of 1.5 at time $t = 0$, and which is worth, at time $t = 1$, i when the state corresponds to ω_i, for all i =1,2,3.
— A put option, P on the asset S with strike $K = 2$, which is worth $3/8$ at time $t = 0$.

We assume that the interest rate is $r = 1/3$.

1. Evaluate the gain G as well as the discounted gain G^* in each state for the strategy which consists of the purchase of one unit of the risky asset and a put option. Is this an arbitrage opportunity ?

2. For each state ω_i, $i = 1, 2, 3$, calculate the corresponding price, i.e. price of the asset that pays 1 when ω_i is realized and 0 if not.
3. We add a fourth state ω_4 where the price of the asset is worth 4, the other parameters remain unchanged. Are there any arbitrage opportunities in this case ? Characterize the set of risk-neutral probabilities.
4. Is this new market complete ?
5. Can we complete this market with a Call with strike $K = 2$? with a Put with strike $K = 4$? with strike $K = 3$? What are the no-arbitrage bounds for the prices of these three assets ?
6. We assume that the market is completed with a Put with strike $K = 3$ and price $7/8$. Calculate the risk-neutral probability in this new market. Calculate the price of a strike call $K = 3$.

EXERCISE 9 - Show that

1. Call prices are non increasing w.r.t to strike
$$K_1 \leq K_2 \quad \Rightarrow \quad \mathrm{Call}(T, K_1) \geq \mathrm{Call}(T, K_2).$$
(Put prices are non decreasing)

2. $\mathrm{Call}_t(T, K)$ and $\mathrm{Put}_t(T, K)$ are Lipschitz w.r.t to the strike, namely :
$$|\mathrm{Call}_t(T, K_1) - \mathrm{Call}_t(T, K_2)| \leq e^{-r(T-t)}|K_1 - K_2|$$
$$|\mathrm{Put}_t(T, K_1) - \mathrm{Put}_t(T, K_2)| \leq e^{-r(T-t)}|K_1 - K_2|$$

3. Call prices are non decreasing w.r.t. maturity : $T_1 \leq T_2$ implies $\mathrm{Call}(T_1, K) \leq \mathrm{Call}(T_2, K)$.

EXERCISE 10 - Capital protected investment

Some investment funds offer their clients a minimum performance guarantee. This type of guarantee can be implemented using options. Suppose that the initial investment is normalized to 1, and that the investor is guaranteed to receive at least K at maturity T (the floor).

1. Give a condition on the value of K so as not to create arbitrage in favor of the investor.

2. The following strategy then makes it possible to respect the constraint while maintaining a potential gain :
   — Invest a fraction λ of the fund in the risky asset S. (we assume $S_0 = 1$.)
   — Use the residual amount to purchase a Put on λS with maturity T and strike K, or, equivalently, λ Puts on S with strike K/λ.
   (a) What is the payoff of the optional part and the value of the fund in T ?
   (b) Give the constraint on λ.

Financial Mathematics.
Tutorial #3: CRR model & Multi-period models (2024)

EXERCICE 1 - The financial market contains a non-risky asset with price process

$$S_0^0 = 1 , \quad S_1^0(\omega_u) = S_1^0(\omega_d) = 1 + R ,$$

and one risky asset ($d = 1$) with price process

$$S_0 = s , \quad S_1(\omega_u) = su , \quad S_1(\omega_d) = sd ,$$

where s, r, u and d are strictly positive with $u > d$.

1. Show that (NA) holds iif $u > 1 + R > d$. (Give a direct proof)
2. Characterize the risk neutral probability.
3. A contingent claim is defined by its payoff $B_u := B(\omega_u)$ and $B_d := B(\omega_d)$. Compute the price p_0 and the quantity of asset ϕ to replicate the payoff B. Comment on the completeness of the market.

EXERCICE 2 - We consider a binomial financial market with two time periods (CRR model) and parameters $d = 0.95$, $u = 1.1$ and $r = 0.05$. Let $S_0 = 95$ be the initial price of the risky asset.

1. Calculate the price at time $t = 0$ of an Asian Call with strike $K = 100$ and maturity $T = 2$.
2. Calculate the price at time $t = 0$ of a call lookback with strike $K = 100$.
3. Calculate the price at time $t = 0$ of an American strike put $K = 100$.

EXERCICE 3 -Convergence of the Binomial model towards the Black Scholes model

Consider a financial market, consisting of a risk-free asset R normalized to $t = 0$, and a risky asset S, traded over the time period $[0, T]$. Divide the time interval $[0, T]$ into n intervals $[t_i^n, t_{i+1}^n]$ with $t_i^n := \frac{iT}{n}$. We place ourselves within the framework of a binomial model with n periods. Let r_n denote the interest rate of the risk-free asset, the value R_t^n of the risk-free asset at times $t = t_i^n$ is then given by:

$$R_{t_i^n}^n = (1 + r_n)^i.$$

We note X_i^n the return of the risky asset between the times t_{i-1}^n and t_i^n . We then have under the historical probability $\mathbb{P}_n$:

$$\mathbb{P}(X_i^n = u_n) = p_n \quad \text{and} \quad \mathbb{P}(X_i^n = d_n) = 1 - p_n.$$

We recall that the vector $(X_1^n, \ldots, X_n^n)$ is a vector of independent random variables. Let r and σ be two positive constants, r_n, d_n and u_n have the following form:

$$r_n = \frac{rT}{n} \qquad d_n = \left(1 + \frac{rT}{n}\right) e^{-\sigma\sqrt{\frac{T}{n}}} \qquad u_n = \left(1 + \frac{rT}{n}\right) e^{\sigma\sqrt{\frac{T}{n}}}.$$

1. Represent the evolution tree of the risky asset in the model.
2. Show that R_T^n converges to e^{rT} as n tends to infinity.

3. Does (NA) hold in this market ?

4. Express the value $S^n_{t^n_i}$ of the risky asset in t^n_i as a function of S_0 and $(X_1, \dots, X_i)$.

5. Give the dynamics of the process X^n under the neutral risk probability $\mathbb{Q}_n$.

   The probability $\mathbb{Q}_n\left(X^n_i = u_n\right)$ will be denoted q_n in the sequel.

6. Check that we have:

$$q_n \xrightarrow[n\to\infty]{} \frac{1}{2} \qquad n\mathbb{E}_{\mathbb{Q}_n}\left[\ln X^n_1\right] \xrightarrow[n\to\infty]{} \left(r - \frac{\sigma^2}{2}\right)T \qquad nVar_{\mathbb{Q}_n}\left[\ln X^n_1\right] \xrightarrow[n\to\infty]{} \sigma^2 T.$$

7. Show using the characteristic functions the convergence to the following law:

$$\sum_{i=1}^{n} \ln X^n_i \xrightarrow[n\to\infty]{law} \mathcal{N}\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right).$$

8. Deduce that:

$$S^n_T \xrightarrow[n\to\infty]{law} S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T} \qquad \text{with } W_T \sim \mathcal{N}(0,T).$$

   The dynamics of the limit is the one assumed in the Black & Scholes model.

9. Write in expectation form the price of a Put with strike K and maturity T in the n-period binomial model.

10. Deduce that the Put price converges when n tends to infinity to:

$$P_0 := Ke^{-rT}\mathcal{N}(-d_2) - S_0\mathcal{N}(-d_1).$$

   With $\mathcal{N}$ the normal distribution function $\mathcal{N}(0,1)$, d_1 and d_2 given by:

$$d_1 := \frac{\ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \qquad \text{and} \qquad d_2 := d_1 - \sigma\sqrt{T}.$$

11. Conclude by obtaining the formula of Black & Scholes giving the price of the Call:

$$C_0 := S_0\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2).$$

Introduction to financial mathematics.
Tutorial #4: Brownian Motion & Stochastic Integration

EXERCICE 1 -[Stopping times]

1. Let τ_1 and τ_2 be two stopping times. Show that the random variables $\tau_1 \wedge \tau_2$, $\tau_1 \vee \tau_2$ and $\tau_1 + \tau_2$ are also stopping times.
2. Let $(\tau_n)_{n\geq 1}$ be a sequence of stopping times . Show that $\sup_n \tau_n$ is a stopping time.

EXERCICE 2 -[Equality of processes]

- Let X and Y be two stochastic processes defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We assume that they have right-continuous trajectories.
  Show that if X is a *modification* of Y then they are indistinguishable.
- Let $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$ and $\mathbb{P} = \lambda$ the lebesgue measure. Define the process X by
$$[0,1] \times \Omega \ni (t,\omega) \mapsto X_t(\omega) = 1_{\{t=\omega\}} \in \{0,1\}.$$
  We also introduce Y to be the constant process equal to 0.
  Is X a modification of Y? Are the two processes indistinguishable?

EXERCICE 3 -[Square integrable martingale] Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space. We consider a square integrable martingale M with continuous sample path.

1. Show that for $u \leq s \leq t$:
$$\mathbb{E}\big[(M_t - M_u)^2|\mathcal{F}_s\big] = \mathbb{E}\big[(M_t - M_s)^2|\mathcal{F}_s\big] + (M_s - M_u)^2. \tag{1}$$
2. Deduce that, for any subdivision π of $[s,t]$, $0 \leq s < t$:
$$\mathbb{E}\big[M_t^2 - M_s^2|\mathcal{F}_s\big] = \mathbb{E}\big[(M_t - M_s)^2|\mathcal{F}_s\big] = \mathbb{E}\left[\sum_{i=1}^{n}(M_{t_i} - M_{t_{i-1}})^2|\mathcal{F}_s\right], \tag{2}$$
   with $t_0 = s$, $t_n = t$.
3. What is the nature of the process $N := M^2$?
4. We assume moreover that $M_0 = 0$ and that M has bounded variation path. Show then that $M = 0$.

EXERCICE 4 -[Martingales]

Let $(B_t)_{t\geq 0}$ be a Brownian motion and $\mathcal{F}$ its natural filtration, show that the following processes are $\mathcal{F}-$martingales:

1. $(B_t)_{t\geq 0}$;
2. $(B_t^2 - t)_{\geq 0}$;
3. $\left(e^{\sigma B_t - \frac{\sigma^2 t}{2}}\right)_{t\geq 0}$, with $\sigma \in \mathbb{R}$, called the geometric Brownian motion.

EXERCICE 5 -[Brownian Motion as a Gaussian process]
Show that:

1. The Brownian motion is a centered Gaussian process with covariance function $c(s,t) = \mathbb{E}[W_s W_t] = t \wedge s$.

2. Conversely, any continuous centered Gaussian process with c as covariance function is a Brownian Motion.

EXERCICE 6 -[Characterisation of Brownian motion]
Let B be a continuous process such that $B_0 = 0$ p.s. and $\mathcal{F}$ its natural filtration. Show that B is a Brownian motion if, and only if, for all $\lambda \in \mathbb{R}$, the complex process M^λ defined by:

$$M_t^\lambda := e^{i\lambda B_t + \frac{\lambda^2 t}{2}}$$

is a $\mathcal{F}$-martingale.

EXERCICE 7 -[Brownian Motions]
Let $(B_t)_{t\geq 0}$ be a Brownian motion. Show that the following processes are also Brownian motions:

1. $\left(\frac{1}{a} B_{a^2 t}\right)_{t\geq 0}$,

2. $(B_{t+t_0} - B_{t_0})_{t\geq 0}$,

3. The process defined by $tB_{1/t}$ for $t > 0$ and extended by 0 to $t = 0$.

EXERCICE 8 -[Brownian bridge]
Let $(B_t)_{t\geq 0}$ be a Brownian motion. We define a new process $Z = (Z_t)_{0\leq t\leq 1}$ by:

$$Z_t = B_t - tB_1.$$

1. Show that Z is an process independent of B_1.

2. Compute the mean function m_t and the covariance function $K(s,t)$ of the process Z.

3. Show that the process defined for all $t \in [0,1]$ by $\tilde{Z}_t := Z_{1-t}$ has the same distribution as Z.

EXERCICE 9 -[Wiener integral] Let f be such that $\int_0^T f^2(t)dt$ is finite. We consider the process $(X_t)_{t\in[0,1]}$ defined by:

$$X_t = \int_0^t f(u)dW_u$$

where $(W_t)_{t\geq 0}$ is a Standard Brownian Motion and $(\mathcal{F}_t)$ its natural filtration.

1. Show that a limit in $L^2(\Omega)$ of a sequence of variables random Gaussian is necessarily Gaussian.

2. Deduce that the process $(X_t)_{t\in[0,1]}$ is a Centered Gaussian process characterized by:

$$\operatorname{cov}(X_t, X_u) = \int_0^{t\wedge u} f^2(s)ds.$$

3. Show that X is a process with independent increments.

4. What is the law of X_1?

EXERCICE 10 -[Martingale property of the stochastic integral]
For some $\phi \in \mathbb{H}^2$, we set $M_t = \int_0^t \phi_s dB_s$, $0 \leq t \leq T$, where B is a Brownian motion. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration of Brownian motion. We recall the result seen in class that the set $\mathcal{E}^2$ (simple random functions) is dense in $\mathbb{H}^2$.

1. Show that $(M_t)_{t\in[0,T]}$ is a square integrable martingale.
2. Show that $N_t := M_t^2 - \langle M \rangle_t$, $t \in [0, T]$ is a martingale.
3. Let A be a non-decreasing continuous and adapted process such that $A_0 = 0$. Show that if the process $Q_t := M_t^2 - A_t$, $t \in [0, T]$, is a martingale then $A = \langle M \rangle$.

Introduction to financial mathematics.
Tutorial #5: Ito Calculus

EXERCICE 1 -[Itô formula]

1. Calculate $\int_0^t W_s dW_s$.
2. Compute the dynamics of $X_t = \frac{W_t^3}{3} - tW_t$.
3. Compute the dynamics of $X_t = xe^{aW_t+bt}$.

EXERCICE 2 - Recall that an Itô process is given by

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s dB_s$$

for $\alpha, \beta \in \mathbb{H}^2$.

Show that the decomposition of an Ito process is unique.

EXERCICE 3 -[Covariation] For two Ito processes X, Y, we define the covaration process by

$$\langle X, Y \rangle_t = \frac{1}{4} \left(\langle X + Y \rangle_t - \langle X - Y \rangle_t \right).$$

1. What is the nature of the process $XY - \langle X, Y \rangle$, when X and Y are square integrable martingales?
2. Show that the following formula holds true:

$$d(XY)_t = X_t dY_t + Y_t dX_t + d\langle X, Y \rangle_t \;.$$

EXERCICE 4 -[Black Scholes SDE] Let B be a Standard Brownian Motion. We consider the Black Scholes differential equation:

$$dS_t = S_t(\mu dt + \sigma dW_t) \quad \text{et} \quad S_0 = x.$$

1. Using Itô's formula, show that the unique solution of this equation is:

$$S_t = xe^{(\mu - \sigma^2/2)t + \sigma W_t}.$$

2. Calculate $\mathbb{E}[S_t]$.
3. Let $u \in C^{1,2}([0, T] \times \mathbb{R}_+)$. Show, using Itô's formula, that

$$du(t, S_t) = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial S} \mu S_t dt + \frac{\partial u}{\partial S} \sigma S_t dW_t + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 u}{\partial S^2} dt.$$

4. For $\alpha \geq 2$, determine the dynamics of S_t^α.
5. Deduce $\mathbb{E}[S_t^\alpha]$ for $\alpha \geq 2$.

EXERCICE 5 -[Representation of PDE solutions] Let $u \in C^{1,2}([0,T) \times \mathbb{R}) \cap C([0,T] \times \mathbb{R})$ be a solution of the heat PDE

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 \leq t < T, \quad x \in \mathbb{R}, \qquad u(T,x) = g(x), \quad x \in \mathbb{R}.$$

We assume that u and $\frac{\partial u}{\partial x}$ have polynomial growth in x: there exist constants constant $C < \infty$ and $p < \infty$ such that

$$|u(t,x)| \leq C(1 + |x|^p), \quad 0 \leq t \leq T, \quad x \in \mathbb{R}.$$

1. Apply Itô's formula to $u(t+s, x + \sigma W_s)$.
2. Deduce that for all $\varepsilon < T - t$,

$$u(t,x) = \mathbb{E}[u(T - \varepsilon, x + \sigma W_{T-t-\varepsilon})].$$

3. Using the dominated convergence theorem, deduce a probabilistic representation for u:

$$u(t,x) = \mathbb{E}[g(x + \sigma W_{T-t})].$$

Introduction to financial mathematics.
Tutorial #6: Continuous time finance

EXERCICE 1 -[Cameron-Martin formula/Girsanov Theorem] Let $(B_t, 0 \le t \le T)$ be a Standard Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t, 0 \le t \le T)$.

Let $\mathbb{Q}$ be an equivalent probability on $(\Omega, \mathcal{F})$ to $\mathbb{P}$. Then, for all $t \le T$, the Radon-Nikodym density of $\mathbb{Q}_{\mathcal{F}_t}$ vs. $\mathbb{P}_{\mathcal{F}_t}$ is denoted Z_t:

$$Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} > 0$$

where $\mathbb{Q}_{\mathcal{F}_t}$, $\mathbb{P}_{\mathcal{F}_t}$ are the Probability restrictions to the σ-algebra $\mathcal{F}_t$.

1. Show that the process Z is a $\mathcal{F}$-martingale positive with respect to $\mathbb{P}$. Calculate $\mathbb{E}[Z_t]$ for all t.
2. Show that under the previous assumptions we have the rule of Bayes: If Y is $\mathcal{F}_t$-measurable,

$$\forall s \le t, \quad \mathbb{E}^{\mathbb{Q}}[Y|\mathcal{F}_s] = \frac{1}{Z_s}\mathbb{E}^{\mathbb{P}}[YZ_t|\mathcal{F}_s]$$

Now let $\mathbb{Q}$ be the probability equivalent to $\mathbb{P}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t := e^{-\lambda B_t - \frac{1}{2}\lambda^2 t}.$$

1. Show that Z is indeed a continuous martingale positive. Calculate $\mathbb{E}[Z_t]$.
2. We recall that if a continuous process verifies

$$\forall \theta \in \mathbb{R}, \quad \mathbb{E}\left[e^{i\theta(W_t - W_s)}|\mathcal{F}_s\right] = e^{-\frac{\theta^2}{2}(t-s)}$$

then W is a Brownian Motion. Show that the process W defined by $W_t = B_t + \sigma t$ is a Brownian Motion under $\mathbb{Q}$.

EXERCICE 2 -[Power Options] In this exercise we consider *a power option of type A*, pay off

$$H_T^A = (S_T^2 - K^2)^+$$

and *a pay-off type B* power option

$$H_T^B = ((S_T - K)^+)^2,$$

under the Black-Scholes model.

1. Using risk-neutral valuation, calculate the price F_t at time t of an asset that pays S_T^2 to time T. Show that F_t^1 follows the Black-Scholes model, calculate its volatility.
2. Show that the type A power option can be seen as a standard call option on the asset F^1. Using the Black-Scholes formula, calculate the price at time t of the type A power option.
3. Calculate the delta (using the formula for deriving a compound function) and describe the dynamic hedging strategy for this option.
4. Show that the pay-off of the type B power option can be expressed in terms of the pay-off of the type A option and the pay-off of the K strike call option.
5. Deduce the price of option type B at time $t < T$.

6. Compute the delta of the type B power option.

7. Deduce the gamma of the type B power option and show that it is bounded by a constant independent of t and S_t. What are the implications for the coverage of this property?

EXERCICE 3 - Let $g : \mathbb{R}_+^* \to \mathbb{R}$ be a bounded continuous function and $T > 0$ a terminal time. We define, for $(t, x) \in [0, T] \times \mathbb{R}_+^*$

$$v(t, x) = \mathbb{E}[g(xe^{-\frac{\sigma^2}{2}(T-t)+\sigma W_{T-t}})] \tag{1}$$

W is a brownian motion and x,σ are positive.

1. Show that $v \in C([0, T] \times \mathbb{R}_+^*)$.

2. Show that $v \in C^{1,2}([0, T) \times \mathbb{R}_+^*)$ and that it has bounded first and second order derivatives on $[0, T - \epsilon] \times \mathbb{R}_+^*$ for every $\epsilon > 0$.

3. Using the expression of the derivatives computed at the previous question, show that v satisfies

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} = 0 \text{ on } [0, T) \times \mathbb{R}_+^*, \quad v(T, x) = g(x). \tag{2}$$

4. Show that any bounded solution $u \in C^{1,2}([0, T) \times \mathbb{R}_+^*) \cap C([0, T] \times \mathbb{R}_+^*)$ with bounded first order derivatives of the PDE (2) writes

$$u(t, x) = \mathbb{E}[g(S_T^{t,x})] \tag{3}$$

where $S_T^{t,x} = x + \int_t^T S_s^{t,x} \sigma \, dW_s$. Conclude.

EXERCICE 4 -[Delta of vanilla option]

1. Recall the Black and Scholes formula for evaluating a Call.

2. Show that for the Call, the Delta is given by $N(d_+)$.

3. Show that for a payoff option g the Delta can be written

$$\Delta = \mathbb{E}^{\mathbb{Q}} \left[e^{-rT} g(S_T) \frac{W_T}{S_0 \sigma T} \right] .$$

EXERCICE 5 -[Geometric Asian option] We consider a financial market made up of a risk-free asset, interest rate $r \geq 0$, and a risky asset S whose dynamics is defined by the Black and Scholes model:

$$\frac{dS_t}{S_t} = bdt + \sigma dW_t^o \ , \ S_0 > 0$$

where $b \in \mathbb{R}$, $\sigma > 0$ are given parameters, and W^o is a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the *usual* conditions. An Asian option on the continuous geometric mean is defined by the payoff at maturity $T > 0$:

$$G := \left(\overline{S}_T - K\right)^+ \text{where } \overline{S}_T = \exp\left(\frac{1}{T}\int_0^T \log(S_t) dt\right).$$

1. Let $\mathbb{Q}$ be the risk-neutral probability and W the Brownian motion associated by Girsanov's theorem. Write the density of $\mathbb{Q}$ and W as a function of W^o.

2. Applying Ito's formula to a well-chosen process, show that $\int_0^T W_t dt = \int_0^T (T-t) dW_t$.
3. Show that

$$\bar{S}_T = \bar{S}_0 e^{rT - \frac{1}{2}\int_0^T \bar{\sigma}(t)^2 dt + \int_0^T \bar{\sigma}(t) dW_t},$$

   where

$$\bar{S}_0 = S_0 e^{-(\frac{r}{2} + \frac{\sigma^2}{12})T} \quad \text{and} \quad \bar{\sigma}(t) = \left(1 - \frac{t}{T}\right)\sigma.$$

4. Recall the risk-neutral valuation formula for the G payoff option.
5. Give the explicit formula for price p_0 at date 0 of the geometric Asian option above.
6. Explain how to construct the hedge portfolio for this option.