Financial Mathematics.
TD 1 & 2, 2024
Introduction to financial derivatives, one-period model.

EXERCISE 1 -Future on a dividend-paying asset

1. Prove that the forward price (term T) of a tradeable asset is

$$K = e^{rT}(S_0 - I),$$

   where S_0 is the spot price, I is the value at $t = 0$ of the dividends (known coupons) paid by the underlying asset between 0 and T (in this case $I > 0$) or the storage cost between 0 and T, which is paid at $t = 0$ (in this case $I < 0$).

2. A treasury bond with a nominal value of EUR 100 and a nominal rate of 8% pays its next coupon in 3 months. Calculate the forward price of this bond with a 6-month maturity if its price today is 106 EUR and the interest rate is 5% per annum.

EXERCISE 2 -Put-Call Parity.
We consider a risky asset whose price at time t is S_t. We assume that the interest rate r is positive. We note $c(t, S_t, T, K)$ (respectively $p(t, S_t, T, K)$) the price of a European *call* (respectively of a *put* European) with strike price K, maturity T and whose underlying asset is S.

1. Show the Put-Call parity property of European options prices :

$$c(0, S_0, T, K) - p(0, S_0, T, K) = S_0 - Ke^{-rT}.$$

2. Deduce that the price of the *call* satisfies

$$(S_0 - Ke^{-rT})^+ \leq c(0, S_0, T, K) \leq S_0.$$

   Show directly (without using 1.) the inequality $S_0 - Ke^{-rT} \leq c(0, S_0, T, K)$.

3. We assume that the risky asset trades at 20 euros, that the price of a European *call* on this asset, with strike price $K = 11$ euros and maturity $T = 1$ year , is 13 euros. We further assume that $r = 9.531\%$. Compute the price of a European *put* with the same characteristics.

4. Show that today's prices $C_0(T, K) = C(0, S_0, T, K)$ and $P_0(T, K) = P(0, S_0, T, K)$ of the *American call* and *American put options* with maturity T and strike price K satisfy

$$C_0(T, K) - P_0(T, K) \leq S_0 - Ke^{-rT}.$$

5. In the Black Scholes model, $S_t = S_0 e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}$, where $(W_t)_{t \geq 0}$ is a Brownian motion. Knowing that W_t is a centered normal random variable with variance t, calculate $\mathbb{E}(e^{-rT}(S_T - K)^+) - \mathbb{E}(e^{-rT}(K - S_T)^+)$. Conclude that the *insurance* approach - which mainly consists in calculating the price as the discounted expectation of the future *payoff* - leads to a contradiction with the European Call-Put parity if $\mu \neq r$.

EXERCISE 3 - Using the same notations as in Exercise 2, show that

1. For any maturity $T > 0$, the price of *call* and *put* are convex at the strike price K.
2. $\forall K > 0$, $\forall 0 \leq T_1 \leq T_2$, $c(0, S_0, T_1, K) \leq c(0, S_0, T_2, Ke^{r(T_2 - T_1)})$.

EXERCISE 4 - Butterfly options.

Given an asset whose price at time T is S_T and strikes $K_1 < K_2 < K_3$, a *butterfly* is a combination of *trading* which is the result of the following net position : a long position on a European *call* with strike price K_1, a long position on a European *call* with strike price K_3, and a position short on two European *calls* with strike price K_2.

1. What is the *payoff* of such an option ? Calculate its price for all $t \leq T$.
2. If K_2 is the middle of the interval $[K_1, K_3]$, show that the *butterfly* can be created by buying and selling options *put* with the different prices of exercise K_1, K_2 and K_3.

NB : *butterfly* options can be bought when the investor believes that the underlying asset will not rise or fall much.

EXERCISE 5 -Effects of dividends on the price of European options.
In this exercise we consider options whose underlying asset pays dividends. Let D be the present value of all the dividends paid over the interval $[0, T]$. Show the relationships :

1. $c_0(T, K) \geq S_0 - D - Ke^{-rT}$,
2. $p_0(T, K) \geq D + Ke^{-rT} - S_0$,
3. (modified Put-Call parity) $c_0(T, K) + D + Ke^{-rT} = p_0(T, K) + S_0$.
4. If the risky asset pays a dividend at time t, show that the price of the *call* remains continuous at t, even if the price of the risky asset is not continuous at t.

EXERCISE 6 - A given asset trades at 95 euros and the European *calls* and *puts* on the given asset, with a strike price of 100 and a maturity of three months, trade respectively at 1.97 euros and 6.57 euros. In one month, the asset will pay a dividend of 1 euro. The prices of the one-month and three-month zero-coupon bonds are 99.60 and 98.60 respectively. Build an arbitrage strategy, if possible.

EXERCISE 7 -American Options We denote by $\text{Call}_t(T, K)$ the price at date t of an European call with strike K and expiry date T, and by $\text{Put}_t(T, K)$ the one of a put with same maturity T and strike K. $\text{CallAmer}_t(T, K)$ and $\text{PutAmer}_t(T, K)$ correspond respectively to the American call and put. We assume that the underlying asset does not pay dividends.

1. Show that for all $t \leq T$,
$$\text{CallAmer}_t(T, K) = \text{Call}_t(T, K).$$
2. Show that for all $t \leq T$,
$$\text{Put}_t(T, K) \leq \text{Amer Put}_t(T, K) \leq \text{Put}_t(T, K) + K(1 - e^{-r(T-t)}).$$

EXERCISE 8 -A one-period model
We consider a one-period market with three states $\omega_1, \omega_2, \omega_3$ and two risky assets :

— An asset S with a value of 1.5 at time $t = 0$, and which is worth, at time $t = 1$, i when the state corresponds to ω_i, for all i =1,2,3.
— A put option, P on the asset S with strike $K = 2$, which is worth 3/8 at time $t = 0$.

We assume that the interest rate is $r = 1/3$.

1. Evaluate the gain G as well as the discounted gain G^* in each state for the strategy which consists of the purchase of one unit of the risky asset and a put option. Is this an arbitrage opportunity ?

2. For each state ω_i, $i = 1, 2, 3$, calculate the corresponding price, i.e. price of the asset that pays 1 when ω_i is realized and 0 if not.
3. We add a fourth state ω_4 where the price of the asset is worth 4, the other parameters remain unchanged. Are there any arbitrage opportunities in this case ? Characterize the set of risk-neutral probabilities.
4. Is this new market complete ?
5. Can we complete this market with a Call with strike $K = 2$? with a Put with strike $K = 4$? with strike $K = 3$? What are the no-arbitrage bounds for the prices of these three assets ?
6. We assume that the market is completed with a Put with strike $K = 3$ and price 7/8. Calculate the risk-neutral probability in this new market. Calculate the price of a strike call $K = 3$.

EXERCISE 9 - Show that

1. Call prices are non increasing w.r.t to strike

$$K_1 \leq K_2 \quad \Rightarrow \quad \text{Call}(T, K_1) \geq \text{Call}(T, K_2).$$

   (Put prices are non decreasing)

2. $\text{Call}_t(T, K)$ and $\text{Put}_t(T, K)$ are Lipschitz w.r.t to the strike, namely :

$$|\text{Call}_t(T, K_1) - \text{Call}_t(T, K_2)| \leq e^{-r(T-t)}|K_1 - K_2|$$
$$|\text{Put}_t(T, K_1) - \text{Put}_t(T, K_2)| \leq e^{-r(T-t)}|K_1 - K_2|$$

3. Call prices are non decreasing w.r.t. maturity : $T_1 \leq T_2$ implies $\text{Call}(T_1, K) \leq \text{Call}(T_2, K)$.

EXERCISE 10 - Capital protected investment

Some investment funds offer their clients a minimum performance guarantee. This type of guarantee can be implemented using options. Suppose that the initial investment is normalized to 1, and that the investor is guaranteed to receive at least K at maturity T (the floor).

1. Give a condition on the value of K so as not to create arbitrage in favor of the investor.

2. The following strategy then makes it possible to respect the constraint while maintaining a potential gain :
   — Invest a fraction λ of the fund in the risky asset S. (we assume $S_0 = 1$.)
   — Use the residual amount to purchase a Put on λS with maturity T and strike K, or, equivalently, λ Puts on S with strike K/λ.

   (a) What is the payoff of the optional part and the value of the fund in T ?
   (b) Give the constraint on λ.