Financial Mathematics.
Tutorial #3: CRR model & Multi-period models (2024)

EXERCICE 1 - The financial market contains a non-risky asset with price process

$$S_0^0 \;=\; 1\;, \quad S_1^0(\omega_u) \;=\; S_1^0(\omega_d) \;= 1+R\;,$$

and one risky asset ($d = 1$) with price process

$$S_0 \;=\; s\;, \quad S_1(\omega_u) \;=\; su\;, \quad S_1(\omega_d) \;=\; sd\;,$$

where s, r, u and d are strictly positive with $u > d$.

1. Show that (NA) holds iif $u \;>\; 1+R \;>\; d$. (Give a direct proof)
2. Characterize the risk neutral probability.
3. A contingent claim is defined by its payoff $B_u := B(\omega_u)$ and $B_d := B(\omega_d)$. Compute the price p_0 and the quantity of asset ϕ to replicate the payoff B. Comment on the completeness of the market.

EXERCICE 2 - We consider a binomial financial market with two time periods (CRR model) and parameters $d = 0.95$, $u = 1.1$ and $r = 0.05$. Let $S_0 = 95$ be the initial price of the risky asset.

1. Calculate the price at time $t = 0$ of an Asian Call with strike $K = 100$ and maturity $T = 2$.
2. Calculate the price at time $t = 0$ of a call lookback with strike $K = 100$.
3. Calculate the price at time $t = 0$ of an American strike put $K = 100$.

EXERCICE 3 -Convergence of the Binomial model towards the Black Scholes model

Consider a financial market, consisting of a risk-free asset R normalized to $t = 0$, and a risky asset S, traded over the time period $[0, T]$. Divide the time interval $[0, T]$ into n intervals $[t_i^n, t_{i+1}^n]$ with $t_i^n := \frac{iT}{n}$. We place ourselves within the framework of a binomial model with n periods. Let r_n denote the interest rate of the risk-free asset, the value R_t^n of the risk-free asset at times $t = t_i^n$ is then given by:

$$R^n_{t_i^n} = (1 + r_n)^i.$$

We note X_i^n the return of the risky asset between the times t_{i-1}^n and t_i^n . We then have under the historical probability $\mathbb{P}_n$:

$$\mathbb{P}(X_i^n = u_n) = p_n \quad \text{and} \quad \mathbb{P}(X_i^n = d_n) = 1 - p_n.$$

We recall that the vector $(X_1^n, \ldots, X_n^n)$ is a vector of independent random variables. Let r and σ be two positive constants, r_n, d_n and u_n have the following form:

$$r_n = \frac{rT}{n} \qquad d_n = \left(1 + \frac{rT}{n}\right) e^{-\sigma\sqrt{\frac{T}{n}}} \qquad u_n = \left(1 + \frac{rT}{n}\right) e^{\sigma\sqrt{\frac{T}{n}}}.$$

1. Represent the evolution tree of the risky asset in the model.
2. Show that R_T^n converges to e^{rT} as n tends to infinity.

3. Does (NA) hold in this market ?
4. Express the value $S^n_{t^n_i}$ of the risky asset in t^n_i as a function of S_0 and $(X_1, \ldots, X_i)$.
5. Give the dynamics of the process X^n under the neutral risk probability $\mathbb{Q}_n$.

   The probability $\mathbb{Q}_n(X^n_i = u_n)$ will be denoted q_n in the sequel.
6. Check that we have:

$$q_n \xrightarrow[n\to\infty]{} \frac{1}{2} \qquad n\mathbb{E}_{\mathbb{Q}_n}\left[\ln X^n_1\right] \xrightarrow[n\to\infty]{} \left(r - \frac{\sigma^2}{2}\right)T \qquad nVar_{\mathbb{Q}_n}\left[\ln X^n_1\right] \xrightarrow[n\to\infty]{} \sigma^2 T.$$

7. Show using the characteristic functions the convergence to the following law:

$$\sum_{i=1}^{n} \ln X^n_i \xrightarrow[n\to\infty]{law} \mathcal{N}\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right).$$

8. Deduce that:

$$S^n_T \xrightarrow[n\to\infty]{law} S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T} \qquad \text{with } W_T \sim \mathcal{N}(0,T).$$

   The dynamics of the limit is the one assumed in the Black & Scholes model.
9. Write in expectation form the price of a Put with strike K and maturity T in the n-period binomial model.
10. Deduce that the Put price converges when n tends to infinity to:

$$P_0 := Ke^{-rT}\mathcal{N}(-d_2) - S_0\mathcal{N}(-d_1).$$

   With $\mathcal{N}$ the normal distribution function $\mathcal{N}(0,1)$, d_1 and d_2 given by:

$$d_1 := \frac{\ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \qquad \text{and} \qquad d_2 := d_1 - \sigma\sqrt{T}.$$

11. Conclude by obtaining the formula of Black & Scholes giving the price of the Call:

$$C_0 := S_0\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2).$$