Introduction to financial mathematics.
Tutorial #4: Brownian Motion & Stochastic Integration

EXERCICE 1 -[Stopping times]

1. Let τ_1 and τ_2 be two stopping times. Show that the random variables $\tau_1 \wedge \tau_2$, $\tau_1 \vee \tau_2$ and $\tau_1 + \tau_2$ are also stopping times.
2. Let $(\tau_n)_{n\geq 1}$ be a sequence of stopping times . Show that $\sup_n \tau_n$ is a stopping time.

EXERCICE 2 -[Equality of processes]

- Let X and Y be two stochastic processes defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We assume that they have right-continuous trajectories.
  Show that if X is a *modification* of Y then they are indistinguishable.
- Let $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$ and $\mathbb{P} = \lambda$ the lebesgue measure. Define the process X by

$$[0,1] \times \Omega \ni (t,\omega) \mapsto X_t(\omega) = 1_{\{t=\omega\}} \in \{0,1\}.$$

  We also introduce Y to be the constant process equal to 0.
  Is X a modification of Y? Are the two processes indistinguishable?

EXERCICE 3 -[Square integrable martingale] Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space. We consider a square integrable martingale M with continuous sample path.

1. Show that for $u \leq s \leq t$:

$$\mathbb{E}\big[(M_t - M_u)^2|\mathcal{F}_s\big] = \mathbb{E}\big[(M_t - M_s)^2|\mathcal{F}_s\big] + (M_s - M_u)^2. \quad (1)$$

2. Deduce that, for any subdivision π of $[s,t]$, $0 \leq s < t$:

$$\mathbb{E}\big[M_t^2 - M_s^2|\mathcal{F}_s\big] = \mathbb{E}\big[(M_t - M_s)^2|\mathcal{F}_s\big] = \mathbb{E}\left[\sum_{i=1}^{n}(M_{t_i} - M_{t_{i-1}})^2|\mathcal{F}_s\right], \quad (2)$$

   with $t_0 = s$, $t_n = t$.
3. What is the nature of the process $N := M^2$?
4. We assume moreover that $M_0 = 0$ and that M has bounded variation path. Show then that $M = 0$.

EXERCICE 4 -[Martingales]

Let $(B_t)_{t\geq 0}$ be a Brownian motion and $\mathcal{F}$ its natural filtration, show that the following processes are $\mathcal{F}-$martingales:

1. $(B_t)_{t\geq 0}$;
2. $(B_t^2 - t)_{\geq 0}$;
3. $\left(e^{\sigma B_t - \frac{\sigma^2 t}{2}}\right)_{t\geq 0}$, with $\sigma \in \mathbb{R}$, called the geometric Brownian motion.

EXERCICE 5 -[Brownian Motion as a Gaussian process]
Show that:

1. The Brownian motion is a centered Gaussian process with covariance function $c(s,t) = \mathbb{E}[W_s W_t] = t \wedge s$.

2. Conversely, any continuous centered Gaussian process with c as covariance function is a Brownian Motion.

EXERCICE 6 -[Characterisation of Brownian motion]

Let B be a continuous process such that $B_0 = 0$ p.s. and $\mathcal{F}$ its natural filtration. Show that B is a Brownian motion if, and only if, for all $\lambda \in \mathbb{R}$, the complex process M^λ defined by:

$$M_t^\lambda := e^{i\lambda B_t + \frac{\lambda^2 t}{2}}$$

is a $\mathcal{F}$-martingale.

EXERCICE 7 -[Brownian Motions]

Let $(B_t)_{t\geq 0}$ be a Brownian motion. Show that the following processes are also Brownian motions:

1. $\left(\frac{1}{a}B_{a^2 t}\right)_{t\geq 0}$,

2. $(B_{t+t_0} - B_{t_0})_{t\geq 0}$,

3. The process defined by $tB_{1/t}$ for $t > 0$ and extended by 0 to $t = 0$.

EXERCICE 8 -[Brownian bridge]

Let $(B_t)_{t\geq 0}$ be a Brownian motion. We define a new process $Z = (Z_t)_{0\leq t\leq 1}$ by:

$$Z_t = B_t - tB_1.$$

1. Show that Z is an process independent of B_1.

2. Compute the mean function m_t and the covariance function $K(s,t)$ of the process Z.

3. Show that the process defined for all $t \in [0,1]$ by $\tilde{Z}_t := Z_{1-t}$ has the same distribution as Z.

EXERCICE 9 -[Wiener integral] Let f be such that $\int_0^T f^2(t)dt$ is finite. We consider the process $(X_t)_{t\in[0,1]}$ defined by:

$$X_t = \int_0^t f(u)dW_u$$

where $(W_t)_{t\geq 0}$ is a Standard Brownian Motion and $(\mathcal{F}_t)$ its natural filtration.

1. Show that a limit in $L^2(\Omega)$ of a sequence of variables random Gaussian is necessarily Gaussian.

2. Deduce that the process $(X_t)_{t\in[0,1]}$ is a Centered Gaussian process characterized by:

$$\mathrm{cov}\,(X_t, X_u) = \int_0^{t\wedge u} f^2(s)ds.$$

3. Show that X is a process with independent increments.

4. What is the law of X_1?

EXERCICE 10 -[Martingale property of the stochastic integral]
For some $\phi \in \mathbb{H}^2$, we set $M_t = \int_0^t \phi_s dB_s$, $0 \le t \le T$, where B is a Brownian motion. We denote by $(\mathcal{F}_t)_{t\ge 0}$ the natural filtration of Brownian motion. We recall the result seen in class that the set $\mathcal{E}^2$ (simple random functions) is dense in $\mathbb{H}^2$.

1. Show that $(M_t)_{t\in[0,T]}$ is a square integrable martingale.
2. Show that $N_t := M_t^2 - \langle M\rangle_t$, $t \in [0,T]$ is a martingale.
3. Let A be a non-decreasing continuous and adapted process such that $A_0 = 0$. Show that if the process $Q_t := M_t^2 - A_t$, $t \in [0,T]$, is a martingale then $A = \langle M\rangle$.