Introduction to financial mathematics.
Tutorial #5: Ito Calculus

EXERCICE 1 -[Itô formula]

1. Calculate $\int_0^t W_s dW_s$.
2. Compute the dynamics of $X_t = \frac{W_t^3}{3} - tW_t$.
3. Compute the dynamics of $X_t = xe^{aW_t+bt}$.

EXERCICE 2 - Recall that an Itô process is given by

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s dB_s$$

for $\alpha, \beta \in \mathbb{H}^2$.

Show that the decomposition of an Ito process is unique.

EXERCICE 3 -[Covariation] For two Ito processes X, Y, we define the covaration process by

$$\langle X, Y \rangle_t = \frac{1}{4} \left(\langle X+Y \rangle_t - \langle X-Y \rangle_t \right).$$

1. What is the nature of the process $XY - \langle X, Y \rangle$, when X and Y are square integrable martingales?
2. Show that the following formula holds true:

$$d(XY)_t = X_t dY_t + Y_t dX_t + d\langle X, Y \rangle_t \ .$$

EXERCICE 4 -[Black Scholes SDE] Let B be a Standard Brownian Motion. We consider the Black Scholes differential equation:

$$dS_t = S_t(\mu dt + \sigma dW_t) \quad \text{et} \quad S_0 = x.$$

1. Using Itô's formula, show that the unique solution of this equation is:

$$S_t = xe^{(\mu - \sigma^2/2)t + \sigma W_t}.$$

2. Calculate $\mathbb{E}[S_t]$.
3. Let $u \in C^{1,2}([0,T] \times \mathbb{R}_+)$. Show, using Itô's formula, that

$$du(t, S_t) = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial S} \mu S_t dt + \frac{\partial u}{\partial S} \sigma S_t dW_t + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 u}{\partial S^2} dt.$$

4. For $\alpha \geq 2$, determine the dynamics of S_t^α.
5. Deduce $\mathbb{E}[S_t^\alpha]$ for $\alpha \geq 2$.

EXERCICE 5 -[Representation of PDE solutions] Let $u \in C^{1,2}([0,T)\times\mathbb{R})\cap C([0,T]\times\mathbb{R})$ be a solution of the heat PDE

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} = 0, \quad 0 \le t < T, \quad x \in \mathbb{R}, \qquad u(T,x) = g(x), \quad x \in \mathbb{R}.$$

We assume that u and $\frac{\partial u}{\partial x}$ have polynomial growth in x: there exist constants constant $C < \infty$ and $p < \infty$ such that

$$|u(t,x)| \le C(1+|x|^p), \quad 0 \le t \le T, \quad x \in \mathbb{R}.$$

1. Apply Itô's formula to $u(t+s, x+\sigma W_s)$.
2. Deduce that for all $\varepsilon < T - t$,

$$u(t,x) = \mathbb{E}[u(T-\varepsilon, x+\sigma W_{T-t-\varepsilon})].$$

3. Using the dominated convergence theorem, deduce a probabilistic representation for u:

$$u(t,x) = \mathbb{E}[g(x+\sigma W_{T-t})].$$