Introduction to financial mathematics.
Tutorial #6: Continuous time finance

EXERCICE 1 -[Cameron-Martin formula/Girsanov Theorem] Let $(B_t, 0 \le t \le T)$ be a Standard Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t, 0 \le t \le T)$.

Let $\mathbb{Q}$ be an equivalent probability on $(\Omega, \mathcal{F})$ to $\mathbb{P}$. Then, for all $t \le T$, the Radon-Nikodym density of $\mathbb{Q}_{\mathcal{F}_t}$ vs. $\mathbb{P}_{\mathcal{F}_t}$ is denoted Z_t:

$$Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} > 0$$

where $\mathbb{Q}_{\mathcal{F}_t}$, $\mathbb{P}_{\mathcal{F}_t}$ are the Probability restrictions to the σ-algebra $\mathcal{F}_t$.

1. Show that the process Z is a $\mathcal{F}$-martingale positive with respect to $\mathbb{P}$. Calculate $\mathbb{E}[Z_t]$ for all t.
2. Show that under the previous assumptions we have the rule of Bayes: If Y is $\mathcal{F}_t$-measurable,

$$\forall s \le t, \quad \mathbb{E}^{\mathbb{Q}}[Y|\mathcal{F}_s] = \frac{1}{Z_s}\mathbb{E}^{\mathbb{P}}[YZ_t|\mathcal{F}_s]$$

Now let $\mathbb{Q}$ be the probability equivalent to $\mathbb{P}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t := e^{-\lambda B_t - \frac{1}{2}\lambda^2 t}.$$

1. Show that Z is indeed a continuous martingale positive. Calculate $\mathbb{E}[Z_t]$.
2. We recall that if a continuous process verifies

$$\forall \theta \in \mathbb{R}, \quad \mathbb{E}\left[e^{i\theta(W_t - W_s)}|\mathcal{F}_s\right] = e^{-\frac{\theta^2}{2}(t-s)}$$

then W is a Brownian Motion. Show that the process W defined by $W_t = B_t + \sigma t$ is a Brownian Motion under $\mathbb{Q}$.

EXERCICE 2 -[Power Options] In this exercise we consider *a power option of type A*, pay off

$$H_T^A = (S_T^2 - K^2)^+$$

and *a pay-off type B* power option

$$H_T^B = ((S_T - K)^+)^2,$$

under the Black-Scholes model.

1. Using risk-neutral valuation, calculate the price F_t at time t of an asset that pays S_T^2 to time T. Show that F_t^1 follows the Black-Scholes model, calculate its volatility.
2. Show that the type A power option can be seen as a standard call option on the asset F^1. Using the Black-Scholes formula, calculate the price at time t of the type A power option.
3. Calculate the delta (using the formula for deriving a compound function) and describe the dynamic hedging strategy for this option.
4. Show that the pay-off of the type B power option can be expressed in terms of the pay-off of the type A option and the pay-off of the K strike call option.
5. Deduce the price of option type B at time $t < T$.

6. Compute the delta of the type B power option.
7. Deduce the gamma of the type B power option and show that it is bounded by a constant independent of t and S_t. What are the implications for the coverage of this property?

EXERCICE 3 - Let $g : \mathbb{R}_+^* \to \mathbb{R}$ be a bounded continuous function and $T > 0$ a terminal time. We define, for $(t, x) \in [0, T] \times \mathbb{R}_+^*$

$$v(t,x) = \mathbb{E}[g(xe^{-\frac{\sigma^2}{2}(T-t)+\sigma W_{T-t}})] \tag{1}$$

W is a brownian motion and x,σ are positive.

1. Show that $v \in C([0, T] \times \mathbb{R}_+^*)$.
2. Show that $v \in C^{1,2}([0, T) \times \mathbb{R}_+^*)$ and that it has bounded first and second order derivatives on $[0, T - \epsilon] \times \mathbb{R}_+^*$ for every $\epsilon > 0$.
3. Using the expression of the derivatives computed at the previous question, show that v satisfies

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} = 0 \text{ on } [0, T) \times \mathbb{R}_+^*, \quad v(T, x) = g(x). \tag{2}$$

4. Show that any bounded solution $u \in C^{1,2}([0, T) \times \mathbb{R}_+^*) \cap C([0, T] \times \mathbb{R}_+^*)$ with bounded first order derivatives of the PDE (2) writes

$$u(t,x) = \mathbb{E}[g(S_T^{t,x})] \tag{3}$$

   where $S_T^{t,x} = x + \int_t^T S_s^{t,x} \sigma \, \mathrm{d}W_s$. Conclude.

EXERCICE 4 -[Delta of vanilla option]

1. Recall the Black and Scholes formula for evaluating a Call.
2. Show that for the Call, the Delta is given by $N(d_+)$.
3. Show that for a payoff option g the Delta can be written

$$\Delta = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} g(S_T) \frac{W_T}{S_0 \sigma T}\right] .$$

EXERCICE 5 -[Geometric Asian option] We consider a financial market made up of a risk-free asset, interest rate $r \geq 0$, and a risky asset S whose dynamics is defined by the Black and Scholes model:

$$\frac{dS_t}{S_t} = bdt + \sigma dW_t^o \ , \ S_0 > 0$$

where $b \in \mathbb{R}$, $\sigma > 0$ are given parameters, and W^o is a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the *usual* conditions. An Asian option on the continuous geometric mean is defined by the payoff at maturity $T > 0$:

$$G := \left(\overline{S}_T - K\right)^+ \text{where } \ \overline{S}_T = \exp\left(\frac{1}{T}\int_0^T \log(S_t) dt\right).$$

1. Let $\mathbb{Q}$ be the risk-neutral probability and W the Brownian motion associated by Girsanov's theorem. Write the density of $\mathbb{Q}$ and W as a function of W^o.

2. Applying Ito's formula to a well-chosen process, show that $\int_0^T W_t dt = \int_0^T (T-t) dW_t$.
3. Show that

$$\bar{S}_T = \bar{S}_0 e^{rT - \frac{1}{2}\int_0^T \bar{\sigma}(t)^2 dt + \int_0^T \bar{\sigma}(t) dW_t},$$

   where

$$\bar{S}_0 = S_0 e^{-(\frac{r}{2} + \frac{\sigma^2}{12})T} \quad \text{and} \quad \bar{\sigma}(t) = \left(1 - \frac{t}{T}\right)\sigma.$$

4. Recall the risk-neutral valuation formula for the G payoff option.
5. Give the explicit formula for price p_0 at date 0 of the geometric Asian option above.
6. Explain how to construct the hedge portfolio for this option.