Week 7, November 17th: Midterm control

Instructor: Yoan Tardy (yoan.tardy@polytechnique.edu)
Tutorial Assistants:

- Nicoleta Cazacu (nicoleta.cazacu@polytechnique.edu)
- Mateo Deangeli Bravo (matdeabra@polytechnique.edu)
- Maxime Marivain (maxime.marivain@universite-paris-saclay.fr)
- Samy Mekkaoui (samy.mekkaoui@polytechnique.edu)

No documents or electronic devices are allowed. Any instance of cheating will lead to a score of zero. Special attention will be given to clarity, precision, and rigorous reasoning throughout the correction.

1 Knowledge Question

1. Compute the moment generating function of a r.v. X following a geometric law with parameter $p \in [0,1]$.

2. Let $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and Y a random variable such that $\mathbb{P}(Y = i) = \alpha_i$ for all $i \in \{1,2,3\}$. Compute and draw its cumulative distribution fonction.

3. Compute $\lim_{n\to\infty} \mathbb{P}(X_n = k)$ for $k \in \mathbb{N}$ where for all $n \geq 1$, X_n is a binomial law with parameters $(n, \lambda/n)$ with $\lambda > 0$.

2 Problem

2.1 Part A

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X \in L^1(\Omega)$.

1. Justify that $\mathbb{P}(X > n) \to 0$ as $n \to \infty$.

2. Deduce that $\mathbb{P}(X < \infty) = 1$.

2.2 Part B

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d real r.v. with $E[X_1^4] < \infty$. We set for all $n \geq 1$, $S_n = \sum_{i=1}^n X_i$. The aim of this part is to show that

$$\mathbb{P}\left[\left\{\omega \in \Omega : \lim_{n\to\infty} S_n(\omega)/n = \mathbb{E}[X_1]\right\}\right] = 1.$$

1. We suppose until question 3. that $\mathbb{E}[X_1] = 0$. Show that $\mathbb{E}[(S_n)^4] = \mathbb{E}[X_1^4]n + 3\mathbb{E}[X_1^2]^2 n(n-1)$.

2. Deduce that $\mathbb{E}\left[\sum_{n\geq 1} \left(\frac{S_n}{n}\right)^4\right] < \infty$.

3. Deduce the result when $\mathbb{E}[X_1] = 0$. (Hint : use Part A).

4. Prove the result in the case where $\mathbb{E}[X_1] \neq 0$.

2.3 Part C

In this part, we take the X_n's such that $\mathbb{P}(X_n = a) = p$ and $\mathbb{P}(X_n = b) = 1 - p$ with $a, b > 0$.

1. Compute $\mathbb{E}[\log(X_1)]$.

2. Deduce that

$$\mathbb{P}\left[\left\{\omega \in \Omega : \lim_{N\to\infty} \left(\prod_{n=1}^{N} X_n\right)^{1/N} = c\right\}\right] = 1,$$

   where you are asked to explicit the determinisitc constant c.

2.4 Part D

In this part, we take the X_n's such that $\mathbb{P}(X_n = k) = p_k$ for all $k \in \mathbb{N}$ where $\sum_{k\in\mathbb{N}} p_k = 1$. Prove that

$$\mathbb{P}\left[\left\{\omega \in \Omega : \lim_{N\to\infty} \frac{\{n \in [[1, N]] : X_n(\omega) = k\}}{N} = c\right\}\right] = 1,$$

where you are asked to explicit the deterministic constant c.

3 Correction

3.1 Part A

. 1. Using the Markov inequality (since $X \in L^1(\Omega)$, we get $\mathbb{P}(X > n) \leq \mathbb{E}[X]/n \to 0$ as $n \to \infty$.

2. According to the monotone continuity of probabilities, we get that $\mathbb{P}(\cap_{n\geq 1}\{X > n\}) = \lim_{n\to\infty} \mathbb{P}(X > n) = 0$. Then it remains to show that

$$\{X = \infty\} = \cap_{n\geq 1}\{X > n\}.$$

The direct inclusion is clear. Then for the reverse inclusion, if $\omega \in \cap_{n\geq 1}\{X > n\}$, then $X(\omega) > n$ for all $n \geq 1$, which implies that $X(\omega) = \infty$.

3.2 Part B

1. By developping the expression and using the linearity of $\mathbb{E}$ we get $\mathbb{E}[S_n^4] = \sum_{i,j,k,\ell} \mathbb{E}[X_i X_j X_k X_\ell]$. Fix i, j, k and $\ell \in [[1, n]]$. Then

- if there exists one of these elements distinct from the other, say i for example,

$$\mathbb{E}[X_i X_j X_k X_\ell] = \mathbb{E}[X_i]\mathbb{E}[X_k X_k X_\ell] = 0$$

  by independance and since $\mathbb{E}[X_i] = 0$.

- if $i = j = k = l$ then $\mathbb{E}[X_i X_j X_k X_\ell] = \mathbb{E}[X_1^4]$. It is clear that there are n such possible scenarios.

- if there are two pairs of indexes giving distinct r.v. (for example $i = j \neq k = \ell$), then $\mathbb{E}[X_i X_j X_k X_\ell] = \mathbb{E}[X_i X_k]\mathbb{E}[X_k X_\ell] = \mathbb{E}[X_1^2]^2$ by independance. To count the number of such possible scenarios, one need to chose the two pairs of indexes that will give the same r.v. there are $\binom{4}{2}$ ways to chose a couple of indexes among 4. but one needs to divide by two because by doing so we double the number of scenarios. Indeed if we chose the couple (i,j), then chosing the pair (k,ℓ) will gave the same result. All in all there are $\binom{4}{2}n(n-1)/2$ scenarios, i.e $3n(n-1)$.

We deduce that

$$\mathbb{E}[S_n^4] = \mathbb{E}[X_1^4]n + 3\mathbb{E}[X_1^2]^2 n(n-1).$$

2. Using Fubini positiv theorem, we get that

$$\mathbb{E}[S_n^4/n^4] = \sum_{n\geq 1} \frac{\mathbb{E}[X_1^4]n + 3\mathbb{E}[X_1^2]^2 n(n-1)}{n^4} \leq C\sum_{n\geq 1}\frac{1}{n^2} < \infty,$$

where C is a positive constant.

3. Using Part A, we see that $\mathbb{P}[\{\omega\in\Omega : \sum_{n\geq 1} S_n^4(\omega)/n < \infty\}] = 1$. Let $\omega_0 \in \Omega : \sum_{n\geq 1} S_n^4(\omega)/n < \infty\}$. Then since $\sum_{n\geq 1} S_n^4(\omega)/n^4 < \infty$, we clearly have $S_n^4(\omega_0)/n^4 \to 0$ as $n\to\infty$. Then,

$$\omega_0 \in \{\omega\in\Omega : \lim_{n\to\infty} S_n(\omega)/n = 0\}.$$

All in all we proved that there exists $A\in\mathcal{F}$ such that

$$A \subset \{\omega\in\Omega : \lim_{n\to\infty} S_n(\omega)/n = 0\},$$

with $\mathbb{P}(A) = 1$, which concludes.

4. Set for all $n\geq 1$, $Y_n = X_n - \mathbb{E}[X_n]$. The sequence $(Y_n)_{n\geq 0}$ is i.i.d with $\mathbb{E}[Y_1] = 0$. Thus we can apply question 3. to this sequence and we obtain

$$\mathbb{P}\Big[\Big\{\omega\in\Omega : \lim_{n\to\infty} n^{-1}\sum_{i=1}^n (X_i - \mathbb{E}[X_i]) = 0\Big\}\Big] = 1.$$

Since

$$\Big\{\omega\in\Omega : \lim_{n\to\infty} n^{-1}\sum_{i=1}^n (X_i - \mathbb{E}[X_i]) = 0\Big\} = \Big\{\omega\in\Omega : \lim_{n\to\infty} n^{-1}\sum_{i=1}^n = \mathbb{E}[X_1]\Big\},$$

the proof is complete.

3.3 Part C

1. $\mathbb{E}[\log(X_1)] = p\log a + (1-p)\log(b)$

2. Since we can write

$$\Big(\prod_{n=1}^N X_n\Big)^{1/N} = \exp\Big(\frac{1}{N}\sum_{n=1}^N \log(X_n)\Big),$$

we conclude using the continuity of the exponential with $c = \exp(p\log a + (1-p)\log b) = a^p b^{1-p}$.

3.4 Part D

We can conclude the same way as Part C with $c = p_k$ since we can write

$$\frac{\{n \in [[1, N]] : X_n = k\}}{N} = \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{X_n = k},$$

the sequence $(\mathbb{1}_{X_n=k})_{n \geq 1}$ being i.i.d with mean p_k.