Week 1, September 23rd: Countability and Permutation of integrals

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Exercise 1. Say if the following sets are countable or not and justify your answer

(i) $A = \left\{\frac{k}{2^n}, k \in \mathbb{Z}, n \in \mathbb{N}\right\}$,

(ii) $B = \left\{(u_n)_{n\geq 0} \in \mathbb{N}^{\mathbb{N}} : \exists N \geq 0, \forall n \geq N, u_n = 0\right\}$

(iii) $C = \left\{(u_n)_{n\geq 0} \in \mathbb{R}^{\mathbb{N}} : \exists N \geq 0, \forall n \geq N, u_n = 0\right\}$

(iv) $D =]0,1]$,

Exercise 2. Using Cantor's diagonal argument, show that $\{0,1\}^{\mathbb{N}}$ is not countable.

Exercise 3. For any set I (countable or not) and any family $(a_i)_{i\in I}$ of nonnegative reels indexed by I, we define

$$\sum_{i\in I} a_i := \sup\left\{\sum_{i\in F} a_i, \quad F \subset I, |F| < \infty\right\}.$$

(i) Show that if $(a_i)_{i\in I}$ is such that $\sum_{i\in I} a_i < \infty$, then $\{i \in I : a_i > 0\}$ is countable. (Hint : consider for all $n \in \mathbb{N}^*$, $D_n = \{i \in I : a_i \geq 1/n\}$.)

(ii) Let $f : [a,b] \to \mathbb{R}$ a non decreasing application with $a, b \in \mathbb{R}$. Deduce from (i) that the set of discontinuity points of f is countable.

(iii) Conclude that if $f : \mathbb{R} \to \mathbb{R}$ is non decreasing, then the set of its discontinuity points is countable.

Exercise 4. Using the Fubini positive theorem, compute

$$I := \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \mathrm{d}x.$$

(Hint : Find the hidden second integral.)

Exercise 5. Let $f : (x,y) \in (\mathbb{R}_+^*)^2 \mapsto e^{-xy}\sin(x)$.

(i) Show that for all $A \geq 1$,

$$\int_0^A \left(\int_0^\infty f(x,y)\mathrm{d}y\right)\mathrm{d}x = \int_0^A \frac{\sin(x)}{x}\mathrm{d}x.$$

(ii) Show that

$$\int_0^\infty \left(\int_0^A f(x,y)\mathrm{d}x\right)\mathrm{d}y = \frac{\pi}{2} + \varepsilon(A),$$

where $\varepsilon(A) \to 0$ as $A \to \infty$. (We admit that for any complexe $z \in \mathbb{C}$, $\int_0^A e^{-zx}\mathrm{d}x = (1 - e^{-zA})/z$.)

(iii) Prove that

$$\int_0^\infty \frac{\sin(x)}{x}\mathrm{d}x = \frac{\pi}{2}.$$