Week 2, September 30th: General probability spaces

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1 Important exercises

Exercise 1. Let $(\Omega, \mathbb{P})$ be a discrete probability space.

(i) Prove that, if A, B are two well chosen events, both $\mathbb{P}[A|B] > \mathbb{P}[A]$ and $\mathbb{P}[A|B] < \mathbb{P}[A]$ can happen.

(ii) Prove that if $(A_n)_{n\in\mathbb{N}}$ is a sequence of events, then $\mathbb{P}[\bigcup_{n\in\mathbb{N}} A_n] \le \sum_{n\in\mathbb{N}} \mathbb{P}[A_n]$.

(iii) Prove that if $(A_n)_{n\in\mathbb{N}}$ is a sequence s.t. $\mathbb{P}[A_n] = 0$ for all $n \in \mathbb{N}$, then $\mathbb{P}[\bigcup_{n\in\mathbb{N}} A_n] = 0$.

(iv) Prove that if $(B_n)_{n\in\mathbb{N}}$ is a sequence s.t. $\mathbb{P}[B_n] = 1$ for all $n \in \mathbb{N}$, then $\mathbb{P}[\bigcap_{n\in\mathbb{N}} B_n] = 1$

Exercise 2. Let A, B, C be three independent events of a discrete probability space $(\Omega, \mathbb{P})$. Show that

(i) $A \cap B$ is independent from C.

(ii) $A \cup B$ is independent from C.

Exercise 3. Let $(\Omega, \mathbb{P})$ be a discrete probability space. Show that if $(B_i)_{i\in I}$ is a finite partition of Ω such that $\mathbb{P}[B_i] > 0 \, \forall i \in I$, then for any $A \subseteq \Omega$

$$\mathbb{P}[A] = \sum_{i\in I} \mathbb{P}[A|B_i]\mathbb{P}[B_i]$$

This is known as the *law of total probability*.

(We recall that $(B_i)_{i\in I}$ is a partition if and only if $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\cup_i B_i = \Omega$.)

Exercise 4. Consider two dices, α and β. α has six regular faces, labeled $\{1, 2, \dots, 6\}$, and β has 12 regular faces, labeled $\{1, 2, \dots, 12\}$. Susie chooses one of the two dices with the same probability and then rolls it n times.

(i) Propose a probability space that models this random experiment

(ii) What is the probability that Susie gets the result 3 at each dice roll?

(iii) What is the probability that Susie gets always the same result at each dice roll?

(iv) What is the conditional probability that Susie picked the dice α given that all dice rolls gave the result 3? Show that this probability is always larger that $\frac{1}{2}$ and tends to 1 as $n \to +\infty$.

Exercise 5. In a town where taxis are either yellow or white with 85% of white taxis, a taxi caused an accident last night. A witness says that the taxi involved was yellow. We know that a witness has a probability of 80% of correctly identifying the color of a taxi at night.

(i) In light of the witness' statement, what is the probability that the taxi involved was actually yellow?

(ii) A second witness confirms that the taxi was yellow. Assuming that the two witnesses do not influence each other, what is the probability that the taxi that caused the accident was yellow in light of the new information?

Exercise 6. Let $(\Omega, \mathbb{P})$ be a discrete probability space and X_1, X_2 two independent random variables defined on it such that

$$\mathbb{P}[X_1 = 1] = \mathbb{P}[X_2 = 1] = \mathbb{P}[X_1 = -1] = \mathbb{P}[X_2 = -1] = 1/2$$

Define the random variables Y_1, Y_2, Y_3 by

$$Y_1 = X_1, \quad Y_2 = X_2, \quad Y_3 = X_1 X_2$$

(i) Show that for any $i, j \in \{1, 2, 3\}$, (Y_i, Y_j) is a family of independent random variables

(ii) Show that (Y_1, Y_2, Y_3) is not a family of independent random variables

Exercise 7. We say that a random variable X is a Bernoulli random variable of parameter p and write $X \sim \mathrm{Be}(p)$ if

$$\mathbb{P}[X = 1] = 1 - \mathbb{P}[X = 0] = p.$$

Moreover, we say that a random variable Y is a binomial of parameters N and p and write $Y \sim \mathrm{Bin}(N, p)$ if

$$\mathbb{P}[Y = k] = \binom{N}{k} p^k (1-p)^{N-k} \quad \forall k = 0, 1 \cdots, N.$$

Let now $(X_1, \cdots, X_N)$ be N independent Bernoulli random variables of parameter p. Show that $Y = \sum_{i=1}^N X_i$ is a binomial of parameters N and p.

Exercise 8. Consider n independent and identically distributed (i.i.d.) random variables $X_1, \ldots, X_n$ with $X_1 \sim \mathrm{Be}(p)$, $p \in (0, 1)$. We may think X_i to be the result of a biased coin toss $(0 = \text{tail}, 1 = \text{head})$. Let S, T be the random variables defined by

$$S = \text{“number of heads obtained”} = \sum_{i=1}^n X_i, \quad T = \text{“time when we get first head”} = \min\{i \le n : X_i = 1\},$$

with the convention that $\inf \emptyset = +\infty$. Compute the joint density $p_{S,T}(\cdot, \cdot)$ and the marginal densities $p_T(\cdot), p_S(\cdot)$. Are S and T independent?

2 Optional exercises

Exercise 9. Let $A_1, \dots, A_n$ be events of a discrete probability space $(\Omega, \mathbb{P})$. Prove inductively that

$$\mathbb{P}\Big[\bigcup_{j=1}^{n} A_j\Big] \geq \sum_{k=1}^{n} \mathbb{P}[A_k] - \sum_{1 \leq i < j \leq n} \mathbb{P}[A_i \cap A_j]$$

This inequality is known as Bonferroni's inequality

Exercise 10. We extract without replacement two balls from an urn with k blue balls and $N-k$ red balls. Propose a probability space that models this random experiment and show that the events"The first ball is blue" and "The second ball is red" are not independent.

Exercise 11. At a lottery, N tickets are sold: m among them are winning tickets. What is the probability that a person who bought r tickets has at least a winning ticket?

Exercise 12. We consider an urn with N blue and N red balls. We also have a set of N bins. We put into the first bin two balls uniformly at random, and leave them there without putting them back in the urn. We go on filling the bins with the same rule: we draw two balls at random from the ones which remain in the urn and then leave them there. What is the probability that each bin has one blue and one red ball?

Exercise 13. We extract three ball from an urn of 50 balls, numbered from 1 to 50. Consider the following random variables

$$\begin{aligned} X &= \{\text{lowest number extracted}\} \\ Z &= \{\text{highest number extracted}\} \\ Y &= \{\text{third number extracted}\} \end{aligned}$$

(i) Compute the marginal densities of X, Y and Z.

(ii) Compute the joint density of (X, Y).