Week 3, October 7th: Expectation and variance

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1 Important exercises

Exercise 1. Let X be a random variable taking values in $\mathbb{N}$. Show that

$$\mathbb{E}[X] = \sum_{k=0}^{+\infty} \mathbb{P}[X > k]$$

Exercise 2. An urn contains $n \geq 1$ white balls and 2 red balls. We extract balls from the urn without replacement and introduce the random variable

$$X = \text{“ number of white balls extracted before finding a red ball”}$$

- Prove that for $k = 0, 1, \dots, n$
$$p_X(k) = \frac{2}{(n+2)(n+1)}(n-k+1)$$
- Compute $\mathbb{E}[X]$

Exercise 3. Let X_1, X_2 be two independent random variables uniformly distributed in $\{1, .., n\}$. We define $Y := \min\{X_1, X_2\}$.

(i) Compute the distribution μ_Y

(ii) Prove that for all $t \leq 1$,
$$\lim_{n \to +\infty} \mathbb{P}[Y \leq tn] = 2t - t^2$$

Exercise 4. Let $(\Omega, \mathbb{P})$ be a discrete p.s.. We recall that for $A \subseteq \Omega$ the corresponding indicator random variable $\mathbb{1}_A$ is defined as follows

$$\mathbb{1}_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

Consider two events $A, B \subseteq \Omega$. Compute the joint density of the random vector $(\mathbb{1}_A, \mathbb{1}_B)$.

Exercise 5. Let X, Y be independent random variables with $X \sim \text{Poi}(\lambda_X)$, $Y \sim \text{Poi}(\lambda_Y)$. Show that $X + Y \sim \text{Poi}(\lambda_X + \lambda_Y)$

Exercise 6. Compute the expectation and the variance of the random variable X in the following cases

(i) X is a Bernoulli random variable of parameter p, i.e. $p_X = \mu$, where μ is the discrete density on $\{0,1\}$ given by

$$\mu(1) = p, \quad \mu(0) = (1-p).$$

We call μ the Bernoulli distribution and denote it $\mathrm{Be}(p)$.

(ii) X is a geometric random variable of parameter p, i.e. $p_X = \mu$, where μ is the discrete density on $\mathbb{N}$ given by

$$\mu(0) = 0, \quad \mu(n) = p(1-p)^{n-1}, n \geq 1.$$

We call μ the geometric distribution and denote it $\mathrm{Geo}(p)$.

(iii) X is a Poisson random variable of parameter λ, i.e. $p_X = \mu$, where μ is the discrete density on $\mathbb{N}$ given by

$$\forall n \in \mathbb{N}, \quad \mu(n) = \exp(-\lambda)\frac{\lambda^n}{n!}$$

We call μ the Poisson distribution and denote it $\mathrm{Poi}(\lambda)$.

Exercise 7. Let $(\Omega, \mathbb{P})$ be a discrete p.s. $X : \Omega \to \mathbb{N} \setminus \{0\}$ be a random variable. Show that the following are equivalent:

1. X is a geometrical random variable, i.e. $X \sim \mathrm{Ge}(p)$ for some $p \in (0,1)$.
2. For any $n \geq 0$, any $k \geq 1$ we have that $\mathbb{P}[X > n] > 0$ and

$$\mathbb{P}[X = n+k \,|\, X > n] = \mathbb{P}[X = k].$$

Hint: Recall the geometric series $\sum_{i=0}^{m} s^i = \frac{1-s^{m+1}}{1-s}$

Exercise 8. Let $X_1, \ldots, X_N$ be real valued random variables defined over the same probability space and in L^2. Show that

(i) The variance of the sum is given by

$$\sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \mathrm{Cov}(X_i, X_j)$$

(ii) Deduce from the first formula that if $X_1, \ldots, X_N$ are independent we have:

$$\mathrm{Var}\Big(\sum_{i=1}^{N} X_i\Big) = \sum_{i=1}^{N} \mathrm{Var}(X_i)$$

(iii) Apply (ii) to compute the variance of a binomial random variable $\mathrm{Bin}(N, p)$.

2 Optional exercises

Exercise 9. Let $p \in (0,1)$ and $n \geq 2$. Consider independent random variables $(Z_1, \ldots, Z_n)$ taking values in $\{-1, 1\}$ with $\mathbb{P}[Z_i = 1] = p$ for all $i \leq n$. Set

$$X = \prod_{i=1}^{n} Z_i$$

(i) Compute $\mathbb{E}[X]$ and deduce from it the distribution of X

(ii) Is X independent from the random vector $(Z_1, \ldots, Z_n)$?

(iii) Is X independent from the random vector $(Z_2, \ldots, Z_n)$?

Exercise 10. Carl and Lucie send out N invitations for their wedding, where $N \mathrm{Poi}(\lambda)$, i.e. N follows the Poisson distribution of parameter λ. Each invitee accepts the invitation with probability $p \in (0,1)$ independently from the other invitees and from the number of invitations. Let X represent the number of people attending the wedding. What is the law of X?