Week 4, October 14th: Inequalities, Caracterisation of laws

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1 Important exercises

Exercise 1. Let X be a random variable taking values in $[0,+\infty)$ and $f:[0,+\infty)\to(0,+\infty)$ an increasing function. Show that for all $\varepsilon>0$

$$\mathbb{P}[X\geq\varepsilon]\leq\frac{\mathbb{E}[f(X)]}{f(\varepsilon)}.$$

Exercise 2. Let $X_1,\ldots,X_n$ be real valued independent random variables, and let M_{X_i} be the respective moment generating functions. Show that for any t such that $M_{X_i}(t)<+\infty$ for all $i=1,\ldots,n$ we have

$$M_{X_1+\ldots+X_n}(t)=\prod_{i=1}^{N}M_{X_i}(t).$$

Use this formula to compute the moment generating function of a binomial random variable $\mathrm{Bin}(N,p)$.

Exercise 3. Let X_1,X_2 be two independent random variables such that $\mu_{X_1}=\mathrm{Geo}(p_1)$, $\mu_{X_2}=\mathrm{Geo}(p_2)$. What is the law of $Y=\min\{X_1,X_2\}$?

Exercise 4. Let X be a Poisson random variable of parameter λ

(i) Compute the moment generating function $M_X(t)$

(ii) Use the former calculation to derive the *Chernoff bounds*

$$\forall t>0,\quad \mathbb{P}[X\geq i]\leq e^{\lambda(e^t-1)}e^{-it}.$$

(iii) Assuming that $i>\lambda$ show that

$$\mathbb{P}[X\geq i]\leq e^{\lambda(i/\lambda-1)}\left(\frac{\lambda}{i}\right)^i$$

(iv) Compute $\mathbb{E}[X^2]$ using the moment generating function M_X.

Exercise 5. In this exercise you should make use of the inequalities seen during the course

(i) Let $X:\Omega\to\mathbb{N}$ be a discrete random variable such that $\mathbb{E}[X]=\mu$ for some $\mu>0$, and let $n\in\mathbb{N}$ be fixed. What can be said about $\mathbb{P}[X\geq n]$? What can be said about $\mathbb{E}[X^4]$?

(ii) Let X be a discrete random variable such that $\mathbb{E}[X]<0$. Suppose that for some $\theta\neq0$, we have $\mathbb{E}[\exp(\theta X)]=1$. Prove that it must be that $\theta\geq0$.

(iii) Let $(\Omega, \mathbb{P})$ be a discrete probability space and $X : \Omega \to \mathbb{N}$ a random variable. Under the assumption that the discrete density $p_X(n)$ is a decreasing function of n show that

$$\forall n \geq 1, \quad p_X(n) \leq \frac{2}{n(n+1)} \mathbb{E}[X]$$

In the proof, you may use the fact that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for all $n \geq 1$.

Exercise 6. Let X be a discrete random variable taking values in $\mathbb{N} \setminus \{0\}$ whose discrete density is given by $p_X(n) = \frac{c_\alpha}{n^{1+\alpha}}$ for all $n \in \mathbb{N}$, where

$$\frac{1}{c_\alpha} = \sum_{n \in \mathbb{N}^*} \frac{1}{n^{1+\alpha}}$$

What are the values of $p \in (0, +\infty)$ for which $X \in L^p$?

2 Optional exercises

Exercise 7. We consider a family of urns $(U_k)_{k \geq 0}$. The urn U_k contains k blue balls and a red ball. Martine picks a random number X whose law is $\mathrm{Poi}(\lambda)$. Next, she picks a ball at random from the urn U_X. We define the event

$$A = \{\text{“Martine picks a red ball”}\}$$

(i) Compute $\mathbb{P}[A | X = k]$ for all $k \in \mathbb{N}$. Deduce from it that $\mathbb{P}[A] = \frac{1 - e^{-\lambda}}{\lambda}$

(ii) Compute $q(k) := \mathbb{P}[X = k | A]$ for all $k \in \mathbb{N}$.

(iii) Recalling that the application $B \mapsto \mathbb{P}[B|A]$ is a probability, the expectation of X with respect to it is a well defined quantity, which we denote $\mathbb{E}[X|A]$. Moreover, we have the formula

$$\mathbb{E}[X|A] = \sum_{k \in \mathbb{N}} k \, \mathbb{P}[X = k | A]$$

Using this formula, compute $\mathbb{E}[X|A]$.

Exercise 8. An investor is faced with the following choices: Either she can invest all of her money in a risky asset that would lead to a random return X that has mean m, or she can put her money into a risk-free venture that will lead to a return of m with probability 1. Suppose that the decision will be made on the basis of maximizing the expected value of $u(R)$, where R is her return and u is her *utility function*, which is assumed to be concave. What choice should the investor make?

Exercise 9. In this exercise, we prove the celebrated *Paley-Zygmund inequality*. Let $X : \Omega \to \mathbb{R}$ be a discrete random variable taking values in $\mathbb{R}$ and such that $\mathbb{E}[X] \geq 0$.

(i) Show that

$$\forall 0 < \lambda < 1, \omega \in \Omega, \quad X(\omega) \leq \lambda \mathbb{E}[X] + X(\omega) \mathbb{1}_{\{X(\omega) > \lambda \mathbb{E}[X]\}}.$$

(ii) Using point (i) prove that for all $0 < \lambda < 1$

$$\mathbb{P}[X > \lambda \mathbb{E}[X]] \geq (1-\lambda)^2 \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$