Week 5, October 21st: Absolute continuous random variables

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1 Important exercises

Exercise 1. Prove that if X is a Bernoulli random variable of parameter $p \in (0,1)$, then X is not an absolutely continuous random variable.

Exercise 2. Let X be a uniformly chosen real number in $[0,2]$. Compute the probability that the equilateral triangle whose side length is X has an area larger or equal than one.

Exercise 3. This exercise is about manipulating exponential random variables. We recall that if X is an exponential random variable of parameter λ, its density is $f_X(t) = \lambda \exp(-\lambda t)\mathbb{1}_{[0,+\infty)}(t)$.

(i) Compute the cumulative distribution function

$$F_X(t) = \mathbb{P}[X \le t]$$

(ii) Let X be an exponential random variable of parameter $\lambda > 0$. What is the law of $Y = \lambda X$ for $\lambda > 0$? (Hint: Look at the cumulative distribution function F_Y)

(iii) Let $X_1, ..., X_n$ be a sequence of independent exponential random variable of parameter $\lambda_i, i = 1, ..., n$. What is the law of $Y = \min(X_1, ..., X_n)$?(Hint: Look at the cumulative distribution function F_Y)

Exercise 4. Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables whose common law is $\mathrm{U}(0,1)$. We set

$$L_n := \min\{X_1, X_2, \dots, X_n\}, \quad Z_n := nL_n.$$

Show that the cumulative distribution function $F_{Z_n}(t)$ converges as $n \to +\infty$ towards a limiting function $F(t)$, that is the cumulative distribution function of a random variable $X \sim \mathrm{Exp}(1)$.

Exercise 5. You arrive at the bus stop at 10 o'clock, knowing that the bus will arrive at a uniformly distributed random time between 10 and 10:30.

(i) What is the probability that you have to wait more than 10 minutes?

(ii) Given that at 10 : 15 the bus has not arrived, what is the probability that you have to wait at least 10 more minutes?

2 Optional exercises

Exercise 6. Starting from floor number $n \in \mathbb{N}$, an elevator travels down a building and sometimes stops at a floor. The elevator cannot go up, and it always goes down at least one floor. The first time, it stops randomly at a floor $p < n$ chosen uniformly. This is repeated at each stop.
Let $A(p,n)$, $p < n$ be the probability that the elevator stops at floor p on its way down.

1. Propose a discrete probability space $(\Omega, \mathbb{P})$ modeling this random experiment.
2. Compute $A(0,n)$, $A(n-1,n)$ and $A(n-2,n)$.
3. For $p \in \{0,\dots,n-2\}$, prove that
$$A(p,n) = \frac{1}{n}\left(1 + \sum_{i=p+1}^{n-1} A(p,i)\right).$$
4. For $p \in \{0,\dots,n-3\}$, prove that $A(p,n) = A(p,n-1)$.
5. For $p \in \{0,\dots,n\}$, let E_p be the random variable equal to 1 if the elevator stops at floor p, and to 0 otherwise. Find the law of E_p. Are the random variables $E_0, E_1, \dots, E_p$ pairwise independent?

Exercise 7.

i) A fire station will be constructed along a road of length $A < \infty$, and the builders would like to determine the ideal place at which to build it. If fires occur at points uniformly chosen on $(0,A)$, where should the station be located so as to minimize the expected distance from the fire? That is, choose a so as to minimize $\mathbb{E}[|X-a|]$ when X is uniformly distributed over $(0,A)$.

ii) Now suppose that the road is of infinite length stretching from point 0 outward to $+\infty$. If the distance of a fire from point 0 is exponentially distributed with rate λ, where should the fire station now be located? That is, we want to minimize $\mathbb{E}[|X-a|]$, where X is now exponential with rate λ.

iii) In the same setting as in question 1), where should the station be located so as to minimize the expected *square* distance $\mathbb{E}[(X-a)^2]$?

Exercise 8. Let $X \sim \mathcal{N}(0,1)$ and Y an independent random variable such that

$$\mathbb{P}(Y=1) = 1 - \mathbb{P}(Y=-1) = p$$

1. Show that $Z = XY$ is a standard Gaussian random variable.
2. Show that $\mathrm{Cov}(X,Z) = 2p-1$
3. Are X and Z independent?