Maximum principle for optimal control problem of non exchangeable mean field systems

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Classical MFC problem

Classical mean-field control (MFC) problem :

$$\inf_{\alpha \in \mathcal{A}} J(\alpha) := \mathbb{E}\Big[\int_0^T f(X_t^\alpha, \mathbb{P}_{X_t^\alpha}, \alpha_t)\mathrm{d}t + g(X_T, \mathbb{P}_{X_T^\alpha})\Big], \tag{1}$$

where $\mathcal{A}$ defines a suitable class of control with controlled state $X^\alpha = (X_t^\alpha)_{t\in[0,T]}$ dynamics given by

$$\begin{aligned} dX_t^\alpha &= b(X_t^\alpha, \mathbb{P}_{X_t^\alpha}, \alpha_t)\mathrm{d}t + \sigma(X_t^\alpha, \mathbb{P}_{X_t^\alpha}, \alpha_t)\mathrm{d}W_t, \\ X_0^\alpha &= \xi, \end{aligned} \tag{2}$$

where the random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ supporting a brownian motion W and an initial random variable ξ.

→ 2 well known methods to study (1)-(2) : DPP and Pontryagin maximum principle.

→ Extend the known *MFG*/*MFC* theory to non exchangeable interactions. A lot of litterature has been developed recently within the graphons theory (see the works of Bayraktar et al. and Aurell et al. for instance) where an agent labeled by $u \in I := [0,1]$ interacts with the other agents through the probability measure

$$\frac{\int_I G(u,v)\mathbb{P}_{X_t^v}(\mathrm{d}x)\mathrm{d}v}{\int_I G(u,v)\mathrm{d}v}) \in \mathcal{P}(\mathbb{R}^d), \quad 0 \leqslant t \leqslant T, \quad u \in I. \tag{3}$$

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→ Extend the framework without specifying the type of interaction. Dynamics are functions of the collection of laws $(\mathbb{P}_{X_t^v})_{v \in I}$.
(see De Crescenzo, Fuhrman, Kharroubi and Pham [2] for the first introduction to this framework).

- Central planner aims to control a system of interacting heterogenous agents :

Non Exchangeable Mean Field SDE

$$dX_t^u = b(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_{v\in I})\mathrm{d}t + \sigma(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_{v\in I})\mathrm{d}W_t^u, \quad 0 \leqslant t \leqslant T, u \in I, \quad (4)$$
$$X_0^u = \xi^u.$$

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$$X_0^u = \xi^u.$$

→ Minimize over a collection of processes $\boldsymbol{\alpha} = (\alpha^u)_{u\in I}$ in a suitable class $\mathcal{A}$ the following cost functional :

Cost Functional

$$J(\boldsymbol{\alpha}) = \int_I \mathbb{E}\Big[\int_0^T f(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_{v\in I})\mathrm{d}t + g(u, X_T^u, (\mathbb{P}_{X_T^v})_{v\in I})\Big]\mathrm{d}u \quad (5)$$

→ Compute $V_0 = J(\boldsymbol{\alpha}^*)$ where $\boldsymbol{\alpha}^*$ is a minimizer of J.

Objectives :

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- Adapt the Pontryagin Maximum Principle to mean field control for non exchangeable mean field systems (NE-MFC) to find necessary and sufficient conditions for an admissible optimal control α.
- Propose an illustration in the Linear Quadratic (LQ) case.

We notice that independantly of our work, Cao and Laurière in [6] study also the Pontryagin Maximum Principle in the context of nonlinear graphon-based interactions, i.e., the coefficients b, σ depend on the collection of laws only through a graphon-weighted measure.

Contents

Following the $NE - MFC$ setting, we need to introduce a suitable space for the collection of measures which we will denote as $L^2(I;\mathcal{P}_2(\mathbb{R}^d)) := L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$.

Definition of $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$

The space $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$ is defined as

$$\Big\{\mu = (\mu^u)_{u\in I} \text{ s.t } I \ni u \mapsto \mu^u \in \mathcal{P}_2(\mathbb{R}^d) \text{ is measurable and} \int_I\int_{\mathbb{R}^d} |x|^2\mu^u(\mathrm{d}x)\mathrm{d}u < \infty\Big\}.$$

- The space $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$ is endowed with the metric :

$$\mathbf{d}(\mu,\nu) := \int_I \mathcal{W}_2(\mu^u,\nu^u)^2\mathrm{d}u, \quad \mu := (\mu^u)_{u\in I}, \quad \nu := (\nu^u)_{u\in I}. \tag{6}$$

- Functions b and σ are now defined on the set $L^2(\mathcal{P}_2(\mathbb{R}^d))$

A derivative in $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$ (1)

(i) Given a function $v : L^2(\mathcal{P}_2(\mathbb{R}^d)) \to \mathbb{R}$, we say that a measurable function

$$\frac{\delta}{\delta m} v : L^2(\mathcal{P}_2(\mathbb{R}^d)) \times I \times \mathbb{R}^d \ \ni \ (\mu, u, x) \ \longmapsto \ \frac{\delta}{\delta m} v(\mu)(u, x) \in \mathbb{R} \tag{7}$$

is the linear functional derivative (or flat derivative) of v if

1. $(\mu, x) \mapsto \frac{\delta}{\delta m} v(\mu)(u, x)$ is continuous from $L^2(\mathcal{P}_2(\mathbb{R}^d)) \times \mathbb{R}^d$ to $\mathbb{R}$ for all $u \in I$;
2. for every compact set $K \subset L^2(\mathcal{P}_2(\mathbb{R}^d))$ there exists a constant $C_K > 0$ such that
$$\left| \frac{\delta}{\delta m} v(\mu)(u, x) \right| \leqslant C_K \, (1 + |x|^2),$$
for all $u \in I$, $x \in \mathbb{R}^d$, $\mu \in K$;
3. we have
$$\begin{aligned} v(\nu) - v(\mu) \quad &= \quad \int_0^1 \langle \frac{\delta}{\delta m} v(\mu + \theta(\nu - \mu)), \nu - \mu \rangle \mathrm{d}\theta \\ &= \quad \int_0^1 \int_I \int_{\mathbb{R}^d} \frac{\delta}{\delta m} v(\mu + \theta(\nu - \mu)(u, x) \, (\nu^u - \mu^u)(\mathrm{d}x) \mathrm{d}u \mathrm{d}\theta \end{aligned}$$
for all $\mu, \nu \in L^2(\mathcal{P}_2(\mathbb{R}^d))$.

A derivative in $L^2(\mathcal{P}_2(\mathbb{R}^d))$ (2)

(ii) We say that the function v admits a continuously differentiable flat derivative if

1. v admits a flat derivative $\frac{\delta}{\delta m}v$ satisfying $x \mapsto \frac{\delta}{\delta m}v(\mu)(u,x)$ is Fréchet differentiable with Fréchet derivative denoted by $x \mapsto \partial\frac{\delta}{\delta m}v(\mu)(u,x)$ for all $(\mu, u) \in L^2(\mathcal{P}_2(\mathbb{R}^d)) \times I$;
2. $(\mu, x) \mapsto \partial\frac{\delta}{\delta m}v(\mu)(u,x)$ is continuous from $L^2(\mathcal{P}_2(\mathbb{R}^d)) \times \mathbb{R}^d$ to $\mathbb{R}$ for all $u \in I$;
3. for every compact set $K \subset L^2(\mathcal{P}_2(\mathbb{R}^d))$ there exists a constant $C_K > 0$ such that

$$\left|\partial\frac{\delta}{\delta m}v(\mu)(u,x)\right| \leqslant C_K\,(1 + |x|^2),$$

for all $u \in I$, $x \in \mathbb{R}^d$, $\mu \in K$.

Gateaux derivative on $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$

Let $f : I \times \mathbb{R}^d \times L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big) \to \mathbb{R}$ assumed to have a continuously differentiable linear functional derivative $\partial \frac{\delta}{\delta m} f$. For $X, Y \in L^2\big(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}^d\big)^I$ such that $(\mathbb{P}_{X^v})_{v\in I}, (\mathbb{P}_{Y^v})_{v\in I} \in L^2(\mathcal{P}_2(\mathbb{R}^d))$ we have

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\big(f(u, x, (\mathbb{P}_{X^v+\epsilon Y^v})_{v\in I}) - f(u, x, (\mathbb{P}_{X^v})_{v\in I})\big) = \int_I \mathbb{E}\big[\partial \frac{\delta}{\delta m} f(u, x, (\mathbb{P}_{X^v})_{v\in I})(\tilde{u}, X^{\tilde{u}}) \cdot Y^{\tilde{u}}\big]\mathrm{d}\tilde{u} \tag{8}$$

Gateaux derivative on $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$

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→ Relation (8) is understood as a calculus of variation on $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$.

A notion of convexity in $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$

Let $f : I \times \mathbb{R}^d \times L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big) \to \mathbb{R}$. f is said to be convex if for every $u \in I$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$, we have :

$$\begin{aligned} f(u, x', \mu') - f(u, x, \mu) \geqslant\ & \partial_x f(u, x, \mu).(x' - x) \\ & + \int_I \mathbb{E}\Big[\partial \frac{\delta}{\delta m} f(u, x, \mu)(\tilde{u}, X^{\tilde{u}}).(X'^{\tilde{u}} - X^{\tilde{u}})\Big] \mathrm{d}\tilde{u}. \end{aligned} \tag{9}$$

where $X'^{\tilde{u}} \sim \mu'^{\tilde{u}}$ and $X^{\tilde{u}} \sim \mu^{\tilde{u}}$.

A notion of convexity in $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$

Let $f : I \times \mathbb{R}^d \times L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big) \to \mathbb{R}$. f is said to be convex if for every $u \in I$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$, we have :

$$f(u, x', \mu') - f(u, x, \mu) \geqslant \partial_x f(u, x, \mu).(x' - x) + \int_I \mathbb{E}\big[\partial \frac{\delta}{\delta m} f(u, x, \mu)(\tilde{u}, X^{\tilde{u}}).(X'^{\tilde{u}} - X^{\tilde{u}})\big] \mathrm{d}\tilde{u}. \tag{9}$$

where $X'^{\tilde{u}} \sim \mu'^{\tilde{u}}$ and $X^{\tilde{u}} \sim \mu^{\tilde{u}}$.

- Can be extended to functions defined on $I \times \mathbb{R}^d \times L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big) \times A$:

$$f(u, x', \mu', a') - f(u, x, \mu, a) \geqslant \partial_x f(u, x, \mu, a) \cdot (a' - a) + \partial_\alpha f(u, x, \mu, a) \cdot (a' - a) + \int_I \mathbb{E}\Big[\partial \frac{\delta}{\delta m} f(u, x, \mu, a)(\tilde{u}, X^{\tilde{u}}).(X'^{\tilde{u}} - X^{\tilde{u}})\Big] \mathrm{d}\tilde{u}. \tag{10}$$

Controlled system **X** dynamics

System dynamics for $\mathbf{X} = (X^u)_{u \in I}$:

$$\begin{cases} dX_t^u = b(u, X_t^u, (\mathbb{P}_{X_t^v})_{v \in I}, \alpha_t^u)\mathrm{d}t + \sigma(u, X_t^u, (\mathbb{P}_{X_t^v})_{v \in I}, \alpha_t^u)\mathrm{d}W_t^u \quad 0 \leqslant t \leqslant T, \\ X_0^u = \xi^u, u \in I. \end{cases} \tag{11}$$

with admissible controls $\boldsymbol{\alpha} = (\alpha^u)_{u \in I}$ are defined as follows. For an arbitrary Borel measurable function function $\alpha : I \times [0, T] \times \mathcal{C}^n_{[0,T]} \times (0,1) \to A$, we define :

$$\alpha_t^u = \alpha(u, t, W^u_{\cdot \wedge t}, U^u), \text{ and } \int_I \int_0^T \mathbb{E}[|\alpha_t^u|^2]\mathrm{d}t\mathrm{d}u < +\infty. \tag{12}$$

Such $\boldsymbol{\alpha}$ is said to be admissible and belongs to $\mathcal{A}$. Moreover, the initial condition $\boldsymbol{\xi} = (\xi^u)_{u \in I}$ is an admissible initial condition if there exists a Borel mesurable function $\xi : I \times (0,1) \to \mathbb{R}^d$ s.t

$$\xi^u = \xi(u, U^u), \text{ and } \int_I \mathbb{E}\left[|\xi^u|^2\right]\mathrm{d}u < +\infty. \tag{13}$$

Study of the controlled system $\mathbf{X}$

Existence and uniqueness for $\mathbf{X}$

Under some standard assumptions on model coefficients b, σ, for an admissible initial condition $\boldsymbol{\xi}$ and an admissible control $\boldsymbol{\alpha} \in \mathcal{A}$, there exists a unique solution to (11) such that there exists a Borel measurable function x defined on $I \times \mathbb{R}^d \times \mathcal{C}^n_{[0,T]} \times (0,1)$ into $\mathbb{R}^d$ with :

$$X^u_t = x(u, t, W^u_{\cdot \wedge t}, U^u), \quad \mathbb{P} \text{ a.s}, \ \ \forall (t,u) \in [0,T] \times I \text{ and } \int_I \mathbb{E}\Big[\sup_{0 \leqslant t \leqslant T} |X^u_t|^2\Big] \mathrm{d}u < +\infty.$$

Study of the controlled system $\mathbf{X}$

Existence and uniqueness for $\mathbf{X}$

Under some standard assumptions on model coefficients b, σ, for an admissible initial condition ξ and an admissible control $\alpha \in \mathcal{A}$, there exists a unique solution to (11) such that there exists a Borel measurable function x defined on $I \times \mathbb{R}^d \times \mathcal{C}^n_{[0,T]} \times (0,1)$ into $\mathbb{R}^d$ with :

$$X^u_t = x(u, t, W^u_{\cdot \wedge t}, U^u), \quad \mathbb{P} \text{ a.s}, \ \forall (t,u) \in [0,T] \times I \text{ and } \int_I \mathbb{E}\Big[\sup_{0 \leqslant t \leqslant T} |X^u_t|^2\Big] \mathrm{d}u < +\infty.$$

$\rightarrow$ This theorem implies the measurability of the mapping $u \mapsto \mathcal{L}(X^u, W^u, U^u)$ which implies under additional standard assumptions f and g that the cost functional (5) is well defined and finite.

Contents

Definition of the Hamiltonian H

The Hamiltonian $\mathbb{R}$-valued function H of the stochastic optimization problem is defined as :

$$H(u, x, \mu, y, z, a) = b(u, x, \mu, a) \cdot y + \sigma(u, x, \mu, a) : z + f(u, x, \mu, a) \tag{14}$$

where $(u, x, \mu, y, z, a) \in I \times \mathbb{R}^d \times L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big) \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \times A$.

$\rightarrow$ Compute an optimality criterion involving the Hamiltonian H assuming differentiability and convexity as defined previously.
$\rightarrow$ In the following, A will denote a convex subset of $\mathbb{R}^m$ for $m \in \mathbb{N}^*$.

We define the 2 following spaces :

$$L^2(I;\mathcal{S}^d) = \{\mathbf{Y} = (Y^u)_{u\in I} : \ Y^u \text{ is } \mathbb{F}^u\text{-adapted and } \int_I \mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} |Y^u_t|^2\Big]\mathrm{d}u < +\infty\}$$

$$L^2(I;\mathbb{H}^{2,d\times n}) = \{\mathbf{Z} = (Z^u)_{u\in I} : Z^u \text{ is } \mathbb{F}^u\text{-adapted and } \int_I \mathbb{E}\Big[\int_0^T |Z^u_t|^2\mathrm{d}t\Big]\mathrm{d}u < +\infty\}$$

Adjoint Equations to **X**

We call adjoint processes of **X** any pair $(\mathbf{Y},\mathbf{Z}) = (Y^u_t, Z^u_t)_{u\in I, t\in[0,T]}$ of processes in $L^2(I;\mathcal{S}^d)\times L^2(I;\mathbb{H}^{2,d\times n})$ satisfying the following conditions

(i) $(\mathbf{Y},\mathbf{Z})$ is solution to the adjoint equations

$$\begin{cases} dY^u_t = -\partial_x H(u, X^u_t, (\mathbb{P}_{X^v_t})_{v\in I}, Y^u_t, Z^u_t, \alpha^u_t)\mathrm{d}t + Z^u_t \mathrm{d}W^u_t \\ \qquad\qquad - \int_I \tilde{\mathbb{E}}\Big[\partial \frac{\delta}{\delta m} H(\tilde{u}, \tilde{X}^{\tilde{u}}_t, (\mathbb{P}_{X^v_t})_{v\in I}, \tilde{Y}^{\tilde{u}}_t, \tilde{Z}^{\tilde{u}}_t, \tilde{\alpha}^{\tilde{u}}_t)(u, X^u_t)\Big]\mathrm{d}\tilde{u}\mathrm{d}t\ , \quad t\in[0,T]\ , \\ Y^u_T = \partial_x g(u, X^u_T, \mathbb{P}_{X^v_T})_{v\in I}) + \int_I \tilde{\mathbb{E}}\Big[\partial \frac{\delta}{\delta m} g(\tilde{u}, \tilde{X}^{\tilde{u}}_T, (\mathbb{P}_{X^v_T})_{v\in I})(u, X^u_T)\Big]\mathrm{d}\tilde{u}\ , \end{cases} \tag{15}$$

for every $u\in I$ where $(\tilde{X},\tilde{Y},\tilde{Z},\tilde{\alpha})$ is an independent copy of (X,Y,Z,α) defined on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$

(ii) There exist Borel functions y and z defined on $I\times[0,T]\times\mathcal{C}^d_{[0,T]}\times(0,1)$ such that

$$Y^u_t = y(u,t,W^u_{\cdot\wedge t}, U^u), \quad \text{and} \quad Z^u_t = z(u,t,W^u_{\cdot\wedge t},U^u), \quad \text{for } t\in[0,T], \mathbb{P}\text{-a.s. and } u\in I.$$

We now state the main results which are obtained under some standard regularity assumptions on b, σ, f and g.

Gâteaux derivative of J

For $\boldsymbol{\beta} \in \mathcal{A}$ such that $\boldsymbol{\alpha} + \epsilon\boldsymbol{\beta} \in \mathcal{A}$ for $\epsilon > 0$ small enough, we have :

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(J(\boldsymbol{\alpha} + \epsilon\boldsymbol{\beta}) - J(\boldsymbol{\alpha})\big) = \int_I \mathbb{E}\Big[\int_0^T \Big(\partial_\alpha H(u, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, Y_t^u, Z_t^u, \alpha_t^u) \cdot \beta_t^u\Big)\mathrm{d}t\Big]\mathrm{d}u$$

where $\mathbf{X}$ is given by (11), $(\mathbf{Y}, \mathbf{Z})$ are given by (15) and the Hamiltonian function H is given by (14).

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where $\mathbf{X}$ is given by (11), $(\mathbf{Y}, \mathbf{Z})$ are given by (15) and the Hamiltonian function H is given by (14).

Necessary condition for optimality of $\boldsymbol{\alpha}$

Moreover, if we assume that H is convex in $a \in A$, that $\boldsymbol{\alpha} = (\alpha_t^u)_{u \in I, 0 \leqslant t \leqslant T}$ is optimal, that $\mathbf{X} = (X_t^u)_{u \in I, 0 \leqslant t \leqslant T}$ is the associated optimal control state given by (11) and that $(\mathbf{Y}, \mathbf{Z}) = (Y_t^u, Z_t^u)_{u \in I, 0 \leqslant t \leqslant T}$ are the associated adjoint processes solving (15), then we have for almost every $u \in I$:

$$\forall a \in A, \quad H(u, X_t^u, (\mathbb{P}_{X_t^v})_{v \in I}, Y_t^u, Z_t^u, \alpha_t^u) \leqslant H(u, X_t^u, (\mathbb{P}_{X_t^v})_{v \in I}, Y_t^u, Z_t^u, a) \quad dt \otimes d\mathbb{P} \text{ a.e} \tag{16}$$

Sufficient condition for optimality of $\boldsymbol{\alpha}$

Let $\boldsymbol{\alpha} = (\alpha^u)_{u \in I} \in \mathcal{A}$, $\mathbf{X}$ the corresponding controlled state process and $(\mathbf{Y}, \mathbf{Z})$ the corresponding adjoint processes. Let also assume that for almost every $u \in I$:

$$(1) \quad \mathbb{R}^d \times L^2(\mathcal{P}_2(\mathbb{R}^d)) \ni (x, \mu) \to g(u, x, \mu) \text{ is convex}$$

$$(2) \quad \mathbb{R}^d \times L^2(\mathcal{P}_2(\mathbb{R}^d)) \times A \ni (x, \mu, a) \to H(u, x, \mu, Y_t^u, Z_t^u, a) \text{ is convex } dt \otimes d\mathbb{P} \text{ a.e}$$

If we assume also following the necessary condition for optimality that for almost every $u \in I$:

$$H(u, X_t^u, (\mathbb{P}_{X_t^v})_{v \in I}, Y_t^u, Z_t^u, \alpha_t^u) = \inf_{\beta \in A} H(u, X_t^u, (\mathbb{P}_{X_t^v})_{v \in I}, Y_t^u, Z_t^u, \beta), \qquad dt \otimes d\mathbb{P} \text{ a.e}$$

Then, $\boldsymbol{\alpha}$ is an optimal control in the sense that $J(\boldsymbol{\alpha}) = \inf_{\boldsymbol{\alpha}' \in \mathcal{A}} J(\boldsymbol{\alpha}')$

- Recall that the convexity property is understood under the definition (9).

Contents

The Pontryagin Maximum principle leads us to study the following collection of fully coupled FBSDE :

Collection of FBSDE system

$$
\begin{cases}
dX_t^u = b(u, t, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, \hat{\alpha}_t^u)\mathrm{d}t + \sigma(u, t, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, \hat{\alpha}_t^u)\mathrm{d}W_t^u, \\
X_0^u = \xi^u, \\
dY_t^u = -\partial_x H(u, t, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, Y_t^u, Z_t^u, \hat{\alpha}_t^u)dt + Z_t^u \mathrm{d}W_t^u \\
\qquad - \int_I \tilde{\mathbb{E}}\left[\partial \frac{\delta}{\delta m} H(\tilde{u}, t, \tilde{X}_t^{\tilde{u}}, (\mathbb{P}_{X_t^v})_{v\in I}, \tilde{Y}_t^{\tilde{u}}, \tilde{Z}_t^{\tilde{u}}, \tilde{\hat{\alpha}}_t^{\tilde{u}})(u, X_t^u)\right] \mathrm{d}\tilde{u}\mathrm{d}t, \\
Y_T^u = \partial_x g(u, X_T^u, (\mathbb{P}_{X_T^v})_{v\in I}) + \int_I \tilde{\mathbb{E}}\left[\partial \frac{\delta}{\delta m} g(\tilde{u}, \tilde{X}_T^{\tilde{u}}, (\mathbb{P}_{X_T^v})_{v\in I})(u, X_T^u)\right] \mathrm{d}\tilde{u}, \\
\hat{\alpha}_t^u = \hat{\mathrm{a}}(u, t, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, Y_t^u, Z_t^u),
\end{cases} \tag{17}
$$

for every $u \in I$, $t \in [0, T]$. $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{\hat{\alpha}})$ is an independant copy of (X, Y, Z, α) defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\tilde{\mathbb{E}}$ denotes the expectation on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$

The Pontryagin Maximum principle leads us to study the following collection of fully coupled FBSDE :

Collection of FBSDE system

$$\begin{cases} dX_t^u = b(u,t,X_t^u,(\mathbb{P}_{X_t^v})_{v\in I},\hat{\alpha}_t^u)\mathrm{d}t + \sigma(u,t,X_t^u,(\mathbb{P}_{X_t^v})_{v\in I},\hat{\alpha}_t^u)\mathrm{d}W_t^u, \\ X_0^u = \xi^u, \\ dY_t^u = -\partial_x H(u,t,X_t^u,(\mathbb{P}_{X_t^v})_{v\in I},Y_t^u,Z_t^u,\hat{\alpha}_t^u)dt + Z_t^u\mathrm{d}W_t^u \\ \qquad - \int_I \tilde{\mathbb{E}}\left[\partial\frac{\delta}{\delta m}H(\tilde{u},t,\tilde{X}_t^{\tilde{u}},(\mathbb{P}_{X_t^v})_{v\in I},\tilde{Y}_t^{\tilde{u}},\tilde{Z}_t^{\tilde{u}},\tilde{\hat{\alpha}}_t^{\tilde{u}})(u,X_t^u)\right]\mathrm{d}\tilde{u}\mathrm{d}t, \\ Y_T^u = \partial_x g(u,X_T^u,(\mathbb{P}_{X_T^v})_{v\in I}) + \int_I \tilde{\mathbb{E}}\left[\partial\frac{\delta}{\delta m}g(\tilde{u},\tilde{X}_T^{\tilde{u}},(\mathbb{P}_{X_T^v})_{v\in I})(u,X_T^u)\right]\mathrm{d}\tilde{u}, \\ \hat{\alpha}_t^u = \hat{\mathrm{a}}(u,t,X_t^u,(\mathbb{P}_{X_t^v})_{v\in I},Y_t^u,Z_t^u), \end{cases} \tag{17}$$

for every $u \in I$, $t \in [0,T]$. $(\tilde{X},\tilde{Y},\tilde{Z},\tilde{\hat{\alpha}})$ is an independant copy of (X,Y,Z,α) defined on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and $\tilde{\mathbb{E}}$ denotes the expectation on the probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$

→ Note that the previous system is indeed a fully coupled FBSDE through the definition of $\hat{\alpha}_t^u$.

→ We are looking for a solution to (17) in a sense to be defined.

Definition of a suitable space of solution $\mathcal{S}$ to the collection of FBSDE (17)

We say that $(X, Y, Z) = (X^u, Y^u, Z^u)_{u \in I}$ belongs to $\mathcal{S}$ if :

- There exists measurable functions x, y and z defined on $I \times [0, T] \times \mathcal{C}^d_{[0,T]} \times [0, 1] \to \mathbb{R}^d$ such that

$$X^u_t = x(u, t, W^u_{\cdot \wedge t}, U^u), \quad Y^u_t = y(u, t, W^u_{\cdot \wedge t}, U^u), \text{ and } Z^u_t = z(u, t, W^u_{\cdot \wedge t}, U^u). \tag{18}$$

- Each process X^u and Y^u are $\mathbb{F}^u$-adapted and continuous and Z^u is $\mathbb{F}^u$-adapted and square integrable.
- The following norm is finite :

$$\|(X, Y, Z)\|_{\mathcal{S}} = \left(\int_I \mathbb{E}\left[\sup_{t \in [0,T]} |X^u_t|^2 + \sup_{t \in [0,T]} |Y^u_t|^2 + \int_0^T |Z^u_t|^2 dt \right] du \right)^{\frac{1}{2}}$$

We say that $(X^u, Y^u, Z^u)_u \in \mathcal{S}$ is a unique solution to (17) if the equations in (17) are satisfied for almost every u. Moreover, we say that the solution is unique if, whenever $(X^u, Y^u, Z^u)_u$, $(\tilde{X}^u, \tilde{Y}^u, \tilde{Z}^u)_u$, the processes (X^u, Y^u, Z^u) and $(\tilde{X}^u, \tilde{Y}^u, \tilde{Z}^u)$ coïncide, up to a $\mathbb{P}$-null set, for almost every $u \in I$.

→ Note that (18) guarantees the measurability of the mapping

$$I \ni u \mapsto \mathcal{L}(X^u, Y^u, Z^u) \in \mathcal{P}_2(\mathcal{C}^d_{[0,T]} \times \mathcal{C}^d_{[0,T]} \times \mathbb{H}^{2, d \times n}_{[0,T]})$$

which justifies the well defined norm $\|.\|_{\mathcal{S}}$.

We now give some assumptions which will ensure existence and uniqueness to the collection of FBSDE (17).

Assumption : Existence and Uniqueness (1)

There exists two constants $L \geqslant 0$ and $\lambda > 0$ such that :

(i) The drift b and the volatility σ are linear in μ, x and α such that :

$$b(u, x, \mu, a) = b_0(u) + \int_I b_2(u, v)\bar{\mu}^v \mathrm{d}v + b_3(u)x + b_4(u)a$$

$$\sigma(u, x, \mu, a) = \sigma_0(u) + \int_I \sigma_2(u, v)\bar{\mu}^v \mathrm{d}v + \sigma_3(u)x + \sigma_4(u)a$$

for some bounded measurable deterministics functions b_0, b_1, b_2, b_3, b_4 with values in $\mathbb{R}^d$, $\mathbb{R}^{d\times d}$, $\mathbb{R}^{d\times d}$, $\mathbb{R}^{d\times m}$ and σ_0, σ_1, σ_2, σ_3 with values in $\mathbb{R}^{d\times n}$, $\mathbb{R}^{(d\times n)\times d}$, $\mathbb{R}^{(d\times n)\times d}$ and $\mathbb{R}^{(d\times n)\times m}$ and where the notation $\bar{\mu}^v = \int_{\mathbb{R}^d} x\mu^v(dx)$.

(ii) The functions f and g satisfy the same assumptions as previously. Moreover, the derivatives of f and g with respect to (x, a) and x respectively are assumed to be L-Lipschitz with respect to (x, a, μ) and (x, μ) respectively where the Lipschitz property in the variable μ is understood in the sense of the distance (6).

(iii) For any $u \in I$, any $x, x' \in \mathbb{R}^d$, any $a, a' \in A$ any $\boldsymbol{\mu} = (\mu^u)_{u\in I}, \boldsymbol{\mu}' = (\mu'^u)_{u\in I} \in L^2(\mathcal{P}_2(\mathbb{R}^d))$, and any $\mathbb{R}^d$ random variables X^u and $X^{',u}$ such that $X^u \sim \mu^u$ and $X^{',u} \sim \mu'^u$, we have :

$$\int_I \mathbb{E}\Big[|\partial \frac{\delta}{\delta m} f(u, x', \mu', a')(\tilde{u}, X'^{\tilde{u}}) - \partial \frac{\delta}{\delta m} f(u, x, \mu, a)(\tilde{u}, X^{\tilde{u}})|^2\Big]\mathrm{d}\tilde{u}$$
$$\leqslant L \left(|x' - x|^2 + |a' - a|^2 + \int_I \mathbb{E}[|X'^u - X^u|^2]\mathrm{d}u \right)$$

Assumption : Existence and Uniqueness (2)

Similarly for g, we have :

$$\int_I \mathbb{E}\Big[|\partial \frac{\delta}{\delta m} g(u, x', \mu')(\tilde{u}, X'^{\tilde{u}}) - \partial \frac{\delta}{\delta m} g(u, x, \mu)(\tilde{u}, X^{\tilde{u}})|^2\Big] \mathrm{d}\tilde{u} \leqslant L\left(|x' - x|^2 + \int_I \mathbb{E}[|X'^{u} - X^{u}|^2] du\right)$$

(iv) The function f satisfies the following convexity property :

$$f(u, x', \mu', a') - f(u, x, \mu, a) - \partial_x f(u, x, \mu, a).(x' - x) - \partial_\alpha f(u, x, \mu, a).(a' - a) - \int_I \mathbb{E}[\partial \frac{\delta}{\delta m} f(u, x, \mu, a)(\tilde{u}, X^{\tilde{u}}).(X'^{\tilde{u}} - X^{\tilde{u}})] \mathrm{d}\tilde{u} \geqslant \lambda |a' - a|^2$$

for all $u \in I$, $(x, \mu, a) \in \mathbb{R}^d \times L^2(\mathcal{P}_2(\mathbb{R}^d)) \times A$ and $(x', \mu', a') \in \mathbb{R}^d \times L^2(\mathcal{P}_2(\mathbb{R}^d)) \times A$, when $X^{\tilde{u}} \sim \mu^{\tilde{u}}$ and $X^{\tilde{u}} \sim \mu^{\tilde{u}}$ and $X'^{\tilde{u}} \sim \mu'^{\tilde{u}}$. We also assume that g is convex in (x, μ) as we did in the sufficient condition in the Pontryagin optimality principle :

$$g(u, x', \mu') - g(u, x, \mu) - \partial_x g(u, x, \mu).(x' - x) - \int_I \mathbb{E}[\partial \frac{\delta}{\delta m} g(u, x, \mu)(\tilde{u}, X^{\tilde{u}}).(X'^{\tilde{u}} - X^{\tilde{u}})] \mathrm{d}\tilde{u} \geqslant 0$$

Theorem : Existence and Uniqueness for a solution to (22)

Under Assumptions 23 and 24 and for any admissible initial condition $\xi = (\xi^u)_{u \in I}$, the collection of forward backward system $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ (22) is uniquely solvable in $\mathcal{S}$.

$\rightarrow$ The proof is based on a classical method of continuation similarly as done in the book of Carmona and Delarue (see [4]).

Contents

Linear graphon models

We consider the following class of models (assuming for sake of simplicity σ constant and $A = \mathbb{R}^m$).

$$\begin{aligned} dX_t^u &= \Big[\beta^u + A^u X_t^u + \int_I G_A(u,v)\mathbb{E}[X_t^v]dv + B^u \alpha_t^u\Big]\mathrm{d}t + \gamma^u \mathrm{d}W_t^u, t \in [0,T] \\ X_0^u &= \xi^u, u \in I, \end{aligned} \tag{19}$$

where $\boldsymbol{\xi} = (\xi^u)_u$ an admissible initial condition and $\boldsymbol{\beta} \in L^\infty(I;\mathbb{R}^d)$, $\boldsymbol{\gamma} \in L^\infty(I;\mathbb{R}^d)$, $\boldsymbol{A} \in L^\infty(I,\mathbb{R}^{d\times d})$, $\boldsymbol{B} \in L^\infty(I;\mathbb{R}^{d\times m})$, $G_A \in L^\infty(I \times I;\mathbb{R}^{d\times d})$.

Quadratic cost functional

$$\begin{aligned} J(\boldsymbol{\alpha}) = \int_I \mathbb{E}\Big[\int_0^T & Q^u\big(X_t^u - \int_I \tilde{G}_Q(u,v)\mathbb{E}[X_t^v]\mathrm{d}v\big)\cdot\big(X_t^u - \int_I \tilde{G}_Q(u,v)\mathbb{E}[X_t^v]\mathrm{d}v\big) + \alpha_t^u \cdot R^u \alpha_t^u \\ & + 2\alpha_t^u \cdot \Gamma^u X_t^u + 2\alpha_t^u \cdot \int_I G_I(u,v)\mathbb{E}[X_t^v]\mathrm{d}v\mathrm{d}t \\ & + H^u\big(X_T^u - \int_I \tilde{G}_H(u,v)\mathbb{E}[X_T^v]\mathrm{d}v\big)\cdot\big(X_T^u - \int_I \tilde{G}_H(u,v)\mathbb{E}[X_T^v]\mathrm{d}v\big)\Big]\mathrm{d}u \end{aligned} \tag{20}$$

Ansatz form for **Y**

We are looking for an ansatz Y_t^u in the following form :

$$Y_t^u = 2\Big(K^u(t)X_t^u + \int_I \bar{K}_t(u,v)\mathbb{E}[X_t^v]\mathrm{d}v\Big) + \Lambda_t^u\Big), \tag{21}$$

where $K \in C^1\big([0,T]; L^\infty(I;\mathbb{S}_+^d)\big)$, $\bar{K} \in C^1\big([0,T], L^2(I \times I;\mathbb{R}^{d\times d})\big)$ and $\Lambda \in C^1\big([0,T]; L^2(I;\mathbb{R}^d)\big)$ are to be determined through infinite dimensional Riccati equations.

→ We inject the form (21) in (17) and we end up with a triangular Riccati system for K, $\bar{K}$ and Λ for which we can prove existence and uniqueness.
→ Finally, we can show existence and uniqueness of the following collection of SDE :

$$\begin{cases} dX_t^u = \Big(\beta^u - B^u(R^u)^{-1}(B^u)^\top\Lambda_t^u + \Big(A^u - B^u(R^u)^{-1}\big((B^u)^\top K_t^u + \Gamma^u\big)\Big)X_t^u \\ \qquad + \int_I \Big(G_A(u,v) - B^u(R^u)^{-1}\big((B^u)^\top\bar{K}_t(u,v) + G_I(u,v)\big)\Big)\mathbb{E}[X_t^v]\mathrm{d}v\Big)\mathrm{d}t + \gamma^u\mathrm{d}W_t^u, \\ X_0^u = \xi^u, \end{cases}$$

Optimal control α in the L-Q case

$$\alpha_t^u = S^u(t)X_t^u + \int_I \bar{S}^{uv}(t)\mathbb{E}[X_t^v]\mathrm{d}v + \Gamma^u(t), \tag{22}$$

where $\boldsymbol{S} = (S^u)_u$, $\bar{\boldsymbol{S}} = (\bar{S}^{uv})_{u,v}$ and $\boldsymbol{\Gamma} = (\Gamma^u)_u$ are deterministic functions, expressed in terms of $K, \bar{K}$ and Λ given by :

$$\begin{cases} S^u(t) = -(R^u)^{-1}\Big((B^u)^\top K_t^u + \Gamma^u\Big), \\ \bar{S}^{uv}(t) = -(R^u)^{-1}\Big((B^u)^\top \bar{K}_t(u,v) + G_I(u,v)\Big) \\ \Gamma^u(t) = -(R^u)^{-1}(B^u)^\top \Lambda_t^u \end{cases}$$

We consider the following model :

$$dX_t^u = [\kappa(X_t^u - \int_I \tilde{G}_\kappa(u,v)\mathbb{E}[X_t^v]dv) + \alpha_t^u]dt + \sigma^u dW_t^u,$$
$$X_0^u = \xi^u, \quad (23)$$

with $\tilde{G}_\kappa$ a bounded, symmetric measurable function from $I \times I$ into $\mathbb{R}$., $\sigma^u > 0$ and $\alpha = (\alpha^u)$ the control process. The initial condition $\xi = (\xi^u)_u$ is assumed to be admissible. The aim of the central bank is then to minimize over α the following cost functional :

Cost functional $J(\boldsymbol{\alpha})$

$$J(\boldsymbol{\alpha}) = \int_I \mathbb{E}\Big[\int_0^T \Big[\eta^u(X_t^u - \int_I \tilde{G}_\eta(u,v)\mathbb{E}[X_t^v]\mathrm{d}v)^2 + q^u\alpha_t^u(X_t^u - \int_I G_q(u,v)\mathbb{E}[X_t^v]\mathrm{d}v) + |\alpha_t^u|^2\Big]\mathrm{d}t$$
$$+ r^u(X_T^u - \int_I \tilde{G}_r(u,v)\mathbb{E}[X_T^v]\mathrm{d}v)^2\Big]du \quad (24)$$

Optimal control form in systemic risk model

Following (22), we end up with the following optimal control form :

$$\hat{\alpha}_t^u = -(K_t^u + \frac{q^u}{2})(X_t^u - \int_I G_Q(u,v)\mathbb{E}[X_t^v]\mathrm{d}v) - \int_I \Big(\bar{K}_t(u,v) + K_t^u G_Q(u,v)\Big)\mathbb{E}[X_t^v]\mathrm{d}v \tag{25}$$

• Setting the coefficients independant of $u \in I$ and $\tilde{G}_\kappa \equiv \tilde{G}_\eta \equiv \tilde{G}_r \equiv G_q \equiv 1$, we recover the classical mean-field result see for systemic risk with $\bar{K}_t \equiv -K_t$ and the optimal control is given by :

$$\hat{\alpha}_t = -(K_t + \frac{q}{2})(X_t - \mathbb{E}[X_t])$$

→ We therefore have the additional term $\int_I \left(\bar{K}_t(u,v) + K_t^u G_Q(u,v)\right) \mathbb{E}[X_t^v]\mathrm{d}v$ compared to the homogeneous case.

Conclusion of our work

- We provide a natural extension to the classical MFC problem in the context of non exchangeable interactions by considering the space $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$.

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- We provide a natural extension to the classical MFC problem in the context of non exchangeable interactions by considering the space $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$.
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- We provide a natural extension to the classical MFC problem in the context of non exchangeable interactions by considering the space $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$.
- It leads to the study of a collection indexed by $u \in I$ of fully coupled FBSDE for which we are able to prove existence and uniqueness under standard assumptions on the model coefficients.
- We provide a semi-analytic form of the LQ graphon model as it leads to the study of infinite dimensional rectangular Riccati equations.

Future works on $NE - MFC$

- Study of the convergence of the N-agent system towards the limit candidate either by showing convergence of value functions of both problems (weak formulation) or convergence of optimal controls (strong formulation). (Partially done in the context of graphons based interactions in the work of Cao and Laurière).
- Study of the $NE - MFC$ with common noise.
- Numerical algorithms in the context of a finite number of players :

  (1) In a model-based setting : Learning optimal controls $\boldsymbol{\alpha} = (\alpha^{1,N}, \ldots, \alpha^{N,N})$ and value function V_N through Deep Learning algorithms.

  (2) In a model-free setting : Learning optimal controls $\boldsymbol{\alpha} = (\alpha^{1,N}, \ldots, \alpha^{N,N})$ and value function V_N through Reinforcement Learning algorithms.

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