Maximum Principle for Non-Exchangeable Mean Field Control Problems with Application to the LQ case

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Clasical MFC problem

The classical mean-field control (MFC) problem can be summarized as the following optimization problem

$$\inf_{\alpha \in \mathcal{A}} J(\alpha) := \mathbb{E}\Big[\int_0^T f(X_t^\alpha, \mathbb{P}_{X_t^\alpha}, \alpha_t)\mathrm{d}t + g(X_T, \mathbb{P}_{X_T^\alpha}) \Big] \tag{1}$$

where $\mathcal{A}$ defines a suitable class of control and where the controlled state $X^\alpha = (X_t^\alpha)_{t\in[0,T]}$ dynamics is given by :

$$\begin{aligned} dX_t^\alpha &= b(X_t^\alpha, \mathbb{P}_{X_t^\alpha}, \alpha_t)\mathrm{d}t + \sigma(X_t^\alpha, \mathbb{P}_{X_t^\alpha}, \alpha_t)\mathrm{d}W_t, \\ X_0^\alpha &= \xi \end{aligned} \tag{2}$$

where the random variables are defined on an abstract filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ supporting a brownian motion W and an initial random variable ξ.

$\rightarrow$ 2 well known methods to study the optimization problem (1)-(2) : Dynamic programming and Pontryagin maximum principle.

→ Extend the known *MFG* theory to non exchangeable interactions. A lot of litterature has been developed recently in the litterature with the Graphons theory (see [6] and [7] for instance) where an agent labeled by $u \in I = [0, 1]$ interacts with the other agents through the probability measure $\frac{\int_I G(u,v)\mathbb{P}_{X_t^v}(\mathrm{d}x)\mathrm{d}v}{\int_I G(u,v)\mathrm{d}v}$).

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→ Our goal is to extend the framework to non exchangeable mean-field systems without specifying the type of interaction and where the dynamics will depend through a term depending on the collection of laws $(\mathbb{P}_{X_t^v})_{v\in I}$ (see De Crescenzo, Fuhrman, Kharroubi and Pham [1] for the first introduction to this framework).

- Central planner aims to control a system of interacting heterogenous agents :

Non Exchangeable Mean Field SDE

$$dX_t^u = b(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_{v\in I})\mathrm{d}t + \sigma(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_{v\in I})\mathrm{d}W_t^u, \quad 0 \leqslant t \leqslant T, u \in I, \tag{3}$$
$$X_0^u = \xi^u.$$

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$$X_0^u = \xi^u.$$

→ Minimize over a collection of processes $\boldsymbol{\alpha} = (\alpha^u)_{u\in I}$ in a suitable class $\mathcal{A}$ the following cost functional :

Cost Functional

$$J(\boldsymbol{\alpha}) = \int_I \mathbb{E}\Big[\int_0^T f(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_{v\in I})\mathrm{d}t + g(u, X_T^u, (\mathbb{P}_{X_T^v})_{v\in I})\Big]\mathrm{d}u \quad (4)$$

→ Compute $V_0 = J(\boldsymbol{\alpha}^*)$ where $\boldsymbol{\alpha}^*$ is a minimizor of J.

Objectives :

- Adapt the Pontryagin Maximum Principle to mean field control for non exchangeable mean field systems (NE-MFC) to find necessary and sufficient conditions for the characterization of an admissible optimal control α.

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- Adapt the Pontryagin Maximum Principle to mean field control for non exchangeable mean field systems (NE-MFC) to find necessary and sufficient conditions for the characterization of an admissible optimal control α.
- Propose an illustration in the Linear Quadratic (LQ) case with an application to systemic risk for heterogeneous banks.

Contents

Following the $NE - MFC$ setting, we need to introduce a suitable space for the collection of measures which we will denote as $L^2(I;\mathcal{P}_2(\mathbb{R}^d)) := L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$.

Definition of $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$:

The space $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$ is defined as follows :

$$\{\boldsymbol{\mu} = (\mu^u)_{u\in I} \text{ s.t } I \ni u \mapsto \mu^u \in \mathcal{P}_2(\mathbb{R}^d) \text{ is measurable and } \int_I \int_{\mathbb{R}^d} |x|^2 \mu^u(\mathrm{d}x)\mathrm{d}u < +\infty\}.$$

• The measurability is understood in the Borel sense given the topological properties of $\mathcal{P}_2(\mathbb{R}^d)$.
• The space $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$ is endowed with the norm :

$$\mathcal{W}_2(\boldsymbol{\mu},\boldsymbol{\nu})^2 = \int_I \mathcal{W}_2(\mu^u,\nu^u)^2 \mathrm{d}u \tag{5}$$

A derivative in $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$ (1)

(i) Given a function $v : L^2(\mathcal{P}_2(\mathbb{R}^d)) \to \mathbb{R}$, we say that a measurable function

$$\frac{\delta}{\delta m} v : L^2(\mathcal{P}_2(\mathbb{R}^d)) \times I \times \mathbb{R}^d \ \ni \ (\mu, u, x) \ \longmapsto \ \frac{\delta}{\delta m} v(\mu)(u, x) \in \mathbb{R} \tag{6}$$

is the linear functional derivative (or flat derivative) of v if

1. $(\mu, x) \mapsto \frac{\delta}{\delta m} v(\mu)(u, x)$ is continuous from $L^2(\mathcal{P}_2(\mathbb{R}^d)) \times \mathbb{R}^d$ to $\mathbb{R}$ for all $u \in I$;
2. for every compact set $K \subset L^2(\mathcal{P}_2(\mathbb{R}^d))$ there exists a constant $C_K > 0$ such that
$$\left| \frac{\delta}{\delta m} v(\mu)(u, x) \right| \leqslant C_K \, (1 + |x|^2),$$
for all $u \in I$, $x \in \mathbb{R}^d$, $\mu \in K$;
3. we have
$$\begin{aligned} v(\nu) - v(\mu) \quad &= \quad \int_0^1 \langle \frac{\delta}{\delta m} v(\mu + \theta(\nu - \mu)), \nu - \mu \rangle \mathrm{d}\theta \\ &= \quad \int_0^1 \int_I \int_{\mathbb{R}^d} \frac{\delta}{\delta m} v(\mu + \theta(\nu - \mu)(u, x) \, (\nu^u - \mu^u)(\mathrm{d}x) \mathrm{d}u \mathrm{d}\theta \end{aligned}$$
for all $\mu, \nu \in L^2(\mathcal{P}_2(\mathbb{R}^d))$.

A derivative in $L^2(\mathcal{P}_2(\mathbb{R}^d))$ (2)

(ii) We say that the function v admits a continuously differentiable flat derivative if

1. v admits a flat derivative $\frac{\delta}{\delta m} v$ satisfying $x \mapsto \frac{\delta}{\delta m} v(\mu)(u, x)$ is Fréchet differentiable with Fréchet derivative denoted by $x \mapsto \partial \frac{\delta}{\delta m} v(\mu)(u, x)$ for all $(\mu, u) \in L^2(\mathcal{P}_2(\mathbb{R}^d)) \times I$;
2. $(\mu, x) \mapsto \partial \frac{\delta}{\delta m} v(\mu)(u, x)$ is continuous from $L^2(\mathcal{P}_2(\mathbb{R}^d)) \times \mathbb{R}^d$ to $\mathbb{R}$ for all $u \in I$;
3. for every compact set $K \subset L^2(\mathcal{P}_2(\mathbb{R}^d))$ there exists a constant $C_K > 0$ such that

$$\left| \partial \frac{\delta}{\delta m} v(\mu)(u, x) \right| \leqslant C_K \left(1 + |x|^2\right),$$

for all $u \in I$, $x \in \mathbb{R}^d$, $\mu \in K$.

Gateaux derivative on $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$

Let $f : I \times \mathbb{R}^d \times L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big) \to \mathbb{R}$ assumed to have a continuously differentiable linear functional derivative $\partial \frac{\delta}{\delta m} f$. For $X, Y \in L^2\big(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}^d\big)^I$ such that $(\mathbb{P}_{X^v})_{v\in I}, (\mathbb{P}_{Y^v})_{v\in I} \in L^2(\mathcal{P}_2(\mathbb{R}^d))$ we have

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\big(f(u, x, (\mathbb{P}_{X^v+\epsilon Y^v})_{v\in I}) - f(u, x, (\mathbb{P}_{X^v})_{v\in I})\big) = \int_I \mathbb{E}\big[\partial \frac{\delta}{\delta m} f(u, x, (\mathbb{P}_{X^v})_{v\in I})(\tilde{u}, X^{\tilde{u}}) \cdot Y^{\tilde{u}}\big]\mathrm{d}\tilde{u} \tag{7}$$

Gateaux derivative on $L^2(\mathcal{P}_2(\mathbb{R}^d))$

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$\to$ The relation (7) is understood as a calculus of variation on $L^2(\mathcal{P}_2(\mathbb{R}^d))$.

A notion of convexity in $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$

Let $f : I \times \mathbb{R}^d \times L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big) \to \mathbb{R}$. f is said to be convex if for every $u \in I$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$, we have :

$$\begin{aligned} f(u, x', \mu') - f(u, x, \mu) \geqslant\ & \partial_x f(u, x, \mu).(x' - x) \\ & + \int_I \mathbb{E}\Big[\partial \frac{\delta}{\delta m} f(u, x, \mu)(\tilde{u}, X^{\tilde{u}}).(X'^{\tilde{u}} - X^{\tilde{u}})\Big] \mathrm{d}\tilde{u}. \end{aligned} \tag{8}$$

A notion of convexity in $L^2(\mathcal{P}_2(\mathbb{R}^d))$

Let $f : I \times \mathbb{R}^d \times L^2(\mathcal{P}_2(\mathbb{R}^d)) \to \mathbb{R}$. f is said to be convex if for every $u \in I$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in L^2(\mathcal{P}_2(\mathbb{R}^d))$, we have :

$$f(u, x', \mu') - f(u, x, \mu) \geqslant \partial_x f(u, x, \mu).(x' - x) + \int_I \mathbb{E}\big[\partial \frac{\delta}{\delta m} f(u, x, \mu)(\tilde{u}, X^{\tilde{u}}).(X'^{\tilde{u}} - X^{\tilde{u}})\big] \mathrm{d}\tilde{u}. \tag{8}$$

• The above convexity definition can be easily extended to the case of functions defined on $I \times \mathbb{R}^d \times L^2(\mathcal{P}_2(\mathbb{R}^d)) \times A$ and reads :

$$f(u, x', \mu', a') - f(u, x, \mu, a) \geqslant \partial_x f(u, x, \mu, a) \cdot (a' - a) + \partial_\alpha f(u, x, \mu, a) \cdot (a' - a) + \int_I \mathbb{E}\Big[\partial \frac{\delta}{\delta m} f(u, x, \mu, a)(\tilde{u}, X^{\tilde{u}}).(X'^{\tilde{u}} - X^{\tilde{u}})\Big] \mathrm{d}\tilde{u}. \tag{9}$$

Definition of the Hamiltonian H :

The Hamiltonian $\mathbb{R}$-valued function H of the stochastic optimization problem is defined as :

$$H(u, x, \mu, y, z, a) = b(u, x, \mu, a) \cdot y + \sigma(u, x, \mu, a) : z + f(u, x, \mu, a) \tag{10}$$

where $(u, x, \mu, y, z, a) \in I \times \mathbb{R}^d \times L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big) \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \times A$.

$\longrightarrow$ Compute an optimality criterion involving the Hamiltonian H assuming differentiability and convexity as defined previously.

Definition of the controlled system $\mathbf{X}$

We assume to be working on a class of mean-field control for non exchangeable systems by considering a collection of controlled state proces $\mathbf{X} = (X^u)_{u\in I}$ described by :

Controlled system $\mathbf{X}$ dynamics

$$\begin{cases} dX_t^u = b(u, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, \alpha_t^u)\mathrm{d}t + \sigma(u, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, \alpha_t^u)\mathrm{d}W_t^u \quad 0 \leqslant t \leqslant T, \\ X_0^u = \xi^u, u \in I. \end{cases} \tag{11}$$

where the admissible control processes $\boldsymbol{\alpha} = (\alpha^u)_{u\in I}$ are defined as follows. For an arbitrary Borel measurable function function $\alpha : I \times [0, T] \times \mathcal{C}^n_{[0,T]} \times (0,1) \to A$, we define :

$$\alpha_t^u = \alpha(u, t, W^u_{\cdot\wedge t}, U^u), \text{ and } \int_I \int_0^T \mathbb{E}[|\alpha_t^u|^2]\mathrm{d}t\mathrm{d}u < +\infty. \tag{12}$$

Such $\boldsymbol{\alpha}$ is said to be admissible and belongs to $\mathcal{A}$. Moreover, the initial condition $\boldsymbol{\xi} = (\xi^u)_{u\in I}$ is an admissible initial condition if there exists a Borel mesurable function $\xi : I \times (0,1) \to \mathbb{R}^d$ s.t

$$\xi^u = \xi(u, U^u), \text{ and } \int_I \mathbb{E}\left[|\xi^u|^2\right]\mathrm{d}u < +\infty. \tag{13}$$

Controlled system $\mathbf{X}$

Existence and uniqueness for $\mathbf{X}$

Under some standard assumptions on model coefficients b, σ, for an admissible initial condition $\boldsymbol{\xi}$ and an admissible control $\boldsymbol{\alpha} \in \mathcal{A}$, there exists a unique solution to (11) such that there exists a Borel measurable function x defined on $I \times \mathbb{R}^d \times \mathcal{C}^n_{[0,T]} \times (0,1)$ into $\mathbb{R}^d$ with :

$$X^u_t = x(u, t, W^u_{\cdot \wedge t}, U^u), \quad \mathbb{P} \text{ a.s}, \ \forall (t,u) \in [0,T] \times I \text{ and } \int_I \mathbb{E}\Big[\sup_{0 \leqslant t \leqslant T} |X^u_t|^2\Big] \mathrm{d}u < +\infty.$$

Controlled system $\mathbf{X}$

Existence and uniqueness for $\mathbf{X}$

Under some standard assumptions on model coefficients b, σ, for an admissible initial condition ξ and an admissible control $\alpha \in \mathcal{A}$, there exists a unique solution to (11) such that there exists a Borel measurable function x defined on $I \times \mathbb{R}^d \times \mathcal{C}^n_{[0,T]} \times (0,1)$ into $\mathbb{R}^d$ with :

$$X_t^u = x(u, t, W^u_{\cdot \wedge t}, U^u), \quad \mathbb{P} \text{ a.s}, \;\; \forall (t,u) \in [0,T] \times I \text{ and } \int_I \mathbb{E}\Big[\sup_{0 \leqslant t \leqslant T} |X_t^u|^2\Big] \mathrm{d}u < +\infty.$$

$\rightarrow$ This theorem implies the measurability of the mapping $u \mapsto \mathcal{L}(X^u, W^u, U^u)$ which implies under additional standard assumptions f and g that the cost functional (4) is well defined and finite.

We define the 2 following spaces :

$$L^2(I;\mathcal{S}^d) = \{\mathbf{Y} = (Y^u)_{u\in I} : \ Y^u \text{ is } \mathbb{F}^u\text{-adapted and } \int_I \mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} |Y_t^u|^2\Big]\mathrm{d}u < +\infty\}$$

$$L^2(I;\mathbb{H}^{2,d\times n}) = \{\mathbf{Z} = (Z^u)_{u\in I} : Z^u \text{ is } \mathbb{F}^u\text{-adapted and } \int_I \mathbb{E}\Big[\int_0^T |Z_t^u|^2\mathrm{d}t\Big]\mathrm{d}u < +\infty\}$$

Adjoint Equations to **X**

We call adjoint processes of **X** any pair $(\mathbf{Y},\mathbf{Z}) = (Y_t^u, Z_t^u)_{u\in I, t\in[0,T]}$ of processes in $L^2(I;\mathcal{S}^d)\times L^2(I;\mathbb{H}^{2,d\times n})$ satisfying the following conditions

(i) $(\mathbf{Y},\mathbf{Z})$ is solution to the adjoint equations

$$\begin{cases} dY_t^u = -\partial_x H(u, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, Y_t^u, Z_t^u, \alpha_t^u)\mathrm{d}t + Z_t^u \mathrm{d}W_t^u \\ \qquad\qquad - \int_I \tilde{\mathbb{E}}\Big[\partial \frac{\delta}{\delta m} H(\tilde{u}, \tilde{X}_t^{\tilde{u}}, (\mathbb{P}_{X_t^v})_{v\in I}, \tilde{Y}_t^{\tilde{u}}, \tilde{Z}_t^{\tilde{u}}, \tilde{\alpha}_t^{\tilde{u}})(u, X_t^u)\Big]\mathrm{d}\tilde{u}\mathrm{d}t\ , \quad t\in[0,T]\ , \\ Y_T^u = \partial_x g(u, X_T^u, \mathbb{P}_{X_T^v})_{v\in I}) + \int_I \tilde{\mathbb{E}}\Big[\partial\frac{\delta}{\delta m} g(\tilde{u}, \tilde{X}_T^{\tilde{u}}, (\mathbb{P}_{X_T^v})_{v\in I})(u, X_T^u)\Big]\mathrm{d}\tilde{u}\ , \end{cases} \tag{14}$$

for every $u\in I$ where $(\tilde{X},\tilde{Y},\tilde{Z},\tilde{\alpha})$ is an independent copy of (X,Y,Z,α) defined on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$

(ii) There exist Borel functions y and z defined on $I\times[0,T]\times\mathcal{C}^d_{[0,T]}\times(0,1)$ such that

$$Y_t^u = y(u,t,W^u_{\cdot\wedge t}, U^u), \quad \text{and} \quad Z_t^u = z(u,t,W^u_{\cdot\wedge t},U^u), \quad \text{for } t\in[0,T],\ \mathbb{P}\text{-a.s. and } u\in I.$$

Contents

We now state the main results which are obtained under some standard regularity assumptions on b, σ, f and g.

Gâteaux derivative of J

For $\boldsymbol{\beta} \in \mathcal{A}$ such that $\boldsymbol{\alpha} + \epsilon\boldsymbol{\beta} \in \mathcal{A}$ for $\epsilon > 0$ small enough, we have :

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(J(\boldsymbol{\alpha} + \epsilon\boldsymbol{\beta}) - J(\boldsymbol{\alpha})\big) = \int_I \mathbb{E}\Big[\int_0^T \Big(\partial_\alpha H(u, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, Y_t^u, Z_t^u, \alpha_t^u) \cdot \beta_t^u\Big)\mathrm{d}t\Big]\mathrm{d}u$$

where $\mathbf{X}$ is given by (11), $(\mathbf{Y}, \mathbf{Z})$ are given by (14) and the Hamiltonian function H is given by (10).

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where $\mathbf{X}$ is given by (11), $(\mathbf{Y}, \mathbf{Z})$ are given by (14) and the Hamiltonian function H is given by (10).

Necessary condition for optimality of $\boldsymbol{\alpha}$

Moreover, if we assume that H is convex in $a \in A$, that $\boldsymbol{\alpha} = (\alpha_t^u)_{u\in I, 0\leqslant t\leqslant T}$ is optimal, that $\mathbf{X} = (X_t^u)_{u\in I, 0\leqslant t\leqslant T}$ is the associated optimal control state given by (11) and that $(\mathbf{Y}, \mathbf{Z}) = (Y_t^u, Z_t^u)_{u\in I, 0\leqslant t\leqslant T}$ are the associated adjoint processes solving (14), then we have for almost every $u \in I$:

$$\forall a \in A, \quad H(u, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, Y_t^u, Z_t^u, \alpha_t^u) \leqslant H(u, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, Y_t^u, Z_t^u, a) \quad dt \otimes d\mathbb{P} \text{ a.e} \tag{15}$$

Sufficient condition for optimality of $\boldsymbol{\alpha}$

Let $\boldsymbol{\alpha} = (\alpha^u)_{u \in I} \in \mathcal{A}$, $\mathbf{X}$ the corresponding controlled state process and $(\mathbf{Y}, \mathbf{Z})$ the corresponding adjoint processes. Let also assume that for almost every $u \in I$:

$$(1) \quad \mathbb{R}^d \times L^2(\mathcal{P}_2(\mathbb{R}^d) \ni (x, \mu) \to g(u, x, \mu) \text{ is convex}$$

$$(2) \quad \mathbb{R}^d \times L^2(\mathcal{P}_2(\mathbb{R}^d) \times A \ni (x, \mu, a) \to H(u, x, \mu, Y_t^u, Z_t^u, a) \text{ is convex } dt \otimes d\mathbb{P} \text{ a.e}$$

If we assume also following the necessary condition for optimality that for almost every $u \in I$:

$$H(u, X_t^u, (\mathbb{P}_{X_t^v})_{v \in I}, Y_t^u, Z_t^u, \alpha_t^u) = \inf_{\beta \in A} H(u, X_t^u, (\mathbb{P}_{X_t^v})_{v \in I}, Y_t^u, Z_t^u, \beta), \qquad dt \otimes d\mathbb{P} \text{ a.e}$$

Then, $\boldsymbol{\alpha}$ is an optimal control in the sense that $J(\boldsymbol{\alpha}) = \inf_{\boldsymbol{\alpha}' \in \mathcal{A}} J(\boldsymbol{\alpha}')$

- Recall that the convexity property is understood under the definition (8).

Contents

The Pontryagin Maximum principle leads us to study the following collection of fully coupled FBSDE :

Collection of FBSDE system

$$\begin{cases} dX_t^u = b(u,t,X_t^u,(\mathbb{P}_{X_t^v})_{v\in I},\hat{\alpha}_t^u)\mathrm{d}t + \sigma(u,t,X_t^u,(\mathbb{P}_{X_t^v})_{v\in I},\hat{\alpha}_t^u)\mathrm{d}W_t^u, \\ X_0^u = \xi^u, \\ dY_t^u = -\partial_x H(u,t,X_t^u,(\mathbb{P}_{X_t^v})_{v\in I},Y_t^u,Z_t^u,\hat{\alpha}_t^u)dt + Z_t^u \mathrm{d}W_t^u \\ \qquad - \int_I \tilde{\mathbb{E}}\left[\partial \frac{\delta}{\delta m} H(\tilde{u},t,\tilde{X}_t^{\tilde{u}},(\mathbb{P}_{X_t^v})_{v\in I},\tilde{Y}_t^{\tilde{u}},\tilde{Z}_t^{\tilde{u}},\tilde{\hat{\alpha}}_t^{\tilde{u}})(u,X_t^u)\right]\mathrm{d}\tilde{u}\mathrm{d}t, \\ Y_T^u = \partial_x g(u,X_T^u,(\mathbb{P}_{X_T^v})_{v\in I}) + \int_I \tilde{\mathbb{E}}\left[\partial\frac{\delta}{\delta m} g(\tilde{u},\tilde{X}_T^{\tilde{u}},(\mathbb{P}_{X_T^v})_{v\in I})(u,X_T^u)\right]\mathrm{d}\tilde{u}, \\ \hat{\alpha}_t^u = \hat{\mathrm{a}}(u,t,X_t^u,(\mathbb{P}_{X_t^v})_{v\in I},Y_t^u,Z_t^u), \end{cases} \tag{16}$$

for every $u \in I$, $t \in [0,T]$. $(\tilde{X},\tilde{Y},\tilde{Z},\tilde{\hat{\alpha}})$ is an independant copy of (X,Y,Z,α) defined on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and $\tilde{\mathbb{E}}$ denotes the expectation on the probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$

The Pontryagin Maximum principle leads us to study the following collection of fully coupled FBSDE :

Collection of FBSDE system

$$\begin{cases} dX_t^u = b(u, t, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, \hat{\alpha}_t^u)\mathrm{d}t + \sigma(u, t, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, \hat{\alpha}_t^u)\mathrm{d}W_t^u, \\ X_0^u = \xi^u, \\ dY_t^u = -\partial_x H(u, t, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, Y_t^u, Z_t^u, \hat{\alpha}_t^u)dt + Z_t^u \mathrm{d}W_t^u \\ \qquad - \int_I \tilde{\mathbb{E}}\left[\partial \frac{\delta}{\delta m} H(\tilde{u}, t, \tilde{X}_t^{\tilde{u}}, (\mathbb{P}_{X_t^v})_{v\in I}, \tilde{Y}_t^{\tilde{u}}, \tilde{Z}_t^{\tilde{u}}, \tilde{\hat{\alpha}}_t^{\tilde{u}})(u, X_t^u)\right] \mathrm{d}\tilde{u}\mathrm{d}t, \\ Y_T^u = \partial_x g(u, X_T^u, (\mathbb{P}_{X_T^v})_{v\in I}) + \int_I \tilde{\mathbb{E}}\left[\partial \frac{\delta}{\delta m} g(\tilde{u}, \tilde{X}_T^{\tilde{u}}, (\mathbb{P}_{X_T^v})_{v\in I})(u, X_T^u)\right] \mathrm{d}\tilde{u}, \\ \hat{\alpha}_t^u = \hat{\mathrm{a}}(u, t, X_t^u, (\mathbb{P}_{X_t^v})_{v\in I}, Y_t^u, Z_t^u), \end{cases} \tag{16}$$

for every $u \in I$, $t \in [0, T]$. $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{\hat{\alpha}})$ is an independant copy of (X, Y, Z, α) defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\tilde{\mathbb{E}}$ denotes the expectation on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$

$\rightarrow$ Note that (20) is indeed a fully coupled FBSDE through the definition of $\hat{\alpha}_t^u$.

$\rightarrow$ We are looking for a solution to (20) in a sense to be defined.

We before introduction a suitable space of solution for (20) which we shall denote by $\mathcal{S}$.

Definition of a suitable space of solution $\mathcal{S}$ to the collection of FBSDE (20)

We say that $(X, Y, Z) = (X^u, Y^u, Z^u)_{u\in I}$ belongs to $\mathcal{S}$ if :

- There exists measurable functions x, y and z defined on $I \times [0, T] \times \mathcal{C}^d_{[0,T]} \times [0, 1] \to \mathbb{R}^d$ such that

$$X^u_t = x(u, t, W^u_{\cdot\wedge t}, Z^u), \quad Y^u_t = y(u, t, W^u_{\cdot\wedge t}, Z^u), \text{ and } Z^u_t = z(u, t, W^u_{\cdot\wedge t}, Z^u).$$

- Each process X^u and Y^u are $\mathbb{F}^u$-adapted and continuous and Z^u is $\mathbb{F}^u$-adapted and square integrable.
- The following norm is finite :

$$\|(X, Y, Z)\|_{\mathcal{S}} = \left(\int_I \mathbb{E}\left[\sup_{t\in[0,T]} |X^u_t|^2 + \sup_{t\in[0,T]} |Y^u_t|^2 + \int_0^T |Z^u_t|^2 dt\right] du\right)^{\frac{1}{2}}$$

We say that $(X^u, Y^u, Z^u)_u \in \mathcal{S}$ is a unique solution to (20) if the equations in (20) are satisfied for almost every u. Moreover, we say that the solution is unique if, whenever $(X^u, Y^u, Z^u)_u$, $(\tilde{X}^u, \tilde{Y}^u, \tilde{Z}^u)_u$, the processes (X^u, Y^u, Z^u) and $(\tilde{X}^u, \tilde{Y}^u, \tilde{Z}^u)$ coïncide, up to a $\mathbb{P}$-null set, for almost every $u \in I$.

$\to$ Note that (17) guarantees the measurability of the mapping

$$I \ni u \mapsto \mathcal{L}(X^u, Y^u, Z^u) \in \mathcal{P}_2(\mathcal{C}^d_{[0,T]} \times \mathcal{C}^d_{[0,T]} \times \mathbb{H}^{2,d\times n}_{[0,T]})$$

which justifies the well defined norm $\|.\|_{\mathcal{S}}$.

We now give the Assumptions which will ensure existence and uniqueness to the collection of FBSDE (20).

Assumption : Existence and Uniqueness (1)

There exists two constants $L \geqslant 0$ and $\lambda > 0$ such that :

(i) The drift b and the volatility σ are linear in μ, x and α such that :

$$b(u, x, \mu, \alpha) = b_0(u) + \int_I b_2(u, v)\bar{\mu}^v \mathrm{d}v + b_3(u)x + b_4(u)\alpha$$

$$\sigma(u, x, \mu, \alpha) = \sigma_0(u) + \int_I \sigma_2(u, v)\bar{\mu}^v \mathrm{d}v + \sigma_3(u)x + \sigma_4(u)\alpha$$

for some bounded measurable deterministics functions b_0, b_1, b_2, b_3, b_4 with values in $\mathbb{R}^d$, $\mathbb{R}^{d\times d}$, $\mathbb{R}^{d\times d}$, $\mathbb{R}^{d\times m}$ and σ_0, σ_1, σ_2, σ_3 with values in $\mathbb{R}^{d\times n}$, $\mathbb{R}^{(d\times n)\times d}$, $\mathbb{R}^{(d\times n)\times d}$ and $\mathbb{R}^{(d\times n)\times m}$ and where the notation $\bar{\mu}^v = \int_{\mathbb{R}^d} x\mu^v(dx)$.

(ii) The functions f and g satisfy the same assumptions as previously. Moreover, the derivatives of f and g with respect to (x, a) and x respectively are assumed to be L-Lipschitz with respect to (x, a, μ) and (x, μ) respectively where the Lipschitz property in the variable μ is understood in the sense of the distance (5).

(iii) For any $u \in I$, any $x, x' \in \mathbb{R}^d$, any $a, a' \in A$ any $\boldsymbol{\mu} = (\mu^u)_{u\in I}, \boldsymbol{\mu}' = (\mu'^u)_{u\in I} \in L^2(\mathcal{P}_2(\mathbb{R}^d))$, and any $\mathbb{R}^d$ random variables X^u and $X'^{,u}$ such that $X^u \sim \mu^u$ and $X'^{,u} \sim \mu'^u$, we have :

$$\int_I \mathbb{E}\Big[|\partial \frac{\delta}{\delta m} f(u, x', \mu', a')(\tilde{u}, X'^{\tilde{u}}) - \partial \frac{\delta}{\delta m} f(u, x, \mu, a)(\tilde{u}, X^{\tilde{u}})|^2\Big]\mathrm{d}\tilde{u}$$
$$\leqslant L\left(|x' - x|^2 + |a' - a|^2 + \int_I \mathbb{E}[|X'^u - X^u|^2]\mathrm{d}u\right)$$

Assumption : Existence and Uniqueness (2)

Similarly for g, we have :

$$\int_I \mathbb{E}\Big[|\partial \frac{\delta}{\delta m} g(u, x', \mu')(\tilde{u}, X'^{\tilde{u}}) - \partial \frac{\delta}{\delta m} g(u, x, \mu)(\tilde{u}, X^{\tilde{u}})|^2\Big] \mathrm{d}\tilde{u} \leqslant L\left(|x' - x|^2 + \int_I \mathbb{E}[|X'^{u} - X^{u}|^2] du\right)$$

(iv) The function f satisfies the following convexity property :

$$f(u, x', \mu', a') - f(u, x, \mu, a) - \partial_x f(u, x, \mu, a).(x' - x) - \partial_\alpha f(u, x, \mu, a).(a' - a) - \int_I \mathbb{E}[\partial \frac{\delta}{\delta m} f(u, x, \mu, a)(\tilde{u}, X^{\tilde{u}}).(X'^{\tilde{u}} - X^{\tilde{u}})] \mathrm{d}\tilde{u} \geqslant \lambda |a' - a|^2$$

for all $u \in I$, $(x, \mu, a) \in \mathbb{R}^d \times L^2(\mathcal{P}_2(\mathbb{R}^d)) \times A$ and $(x', \mu', a') \in \mathbb{R}^d \times L^2(\mathcal{P}_2(\mathbb{R}^d)) \times A$, when $X^{\tilde{u}} \sim \mu^{\tilde{u}}$ and $X^{\tilde{u}} \sim \mu^{\tilde{u}}$ and $X'^{\tilde{u}} \sim \mu'^{\tilde{u}}$. We also assume that g is convex in (x, μ) as we did in the sufficient condition in the Pontryagin optimality principle :

$$g(u, x', \mu') - g(u, x, \mu) - \partial_x g(u, x, \mu).(x' - x) - \int_I \mathbb{E}[\partial \frac{\delta}{\delta m} g(u, x, \mu)(\tilde{u}, X^{\tilde{u}}).(X'^{\tilde{u}} - X^{\tilde{u}})] \mathrm{d}\tilde{u} \geqslant 0$$

Theorem : Existence and Uniqueness for a solution to (20)

Under Assumptions 23 and 24 and for any admissible initial condition $\xi = (\xi^u)_{u \in I}$, the collection of forward backward system $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ (20) is uniquely solvable in $\mathcal{S}$.

Contents

LQ Graphon Models

We consider the following class of models (assuming for sake of simplicity σ constant and $A \subset \mathbb{R}^m$).

$$\begin{aligned} dX_t^u &= \Big[\beta^u + A^u X_t^u + \int_I G_A(u,v)\mathbb{E}[X_t^v]dv + B^u \alpha_t^u\Big]\mathrm{d}t + \gamma^u \mathrm{d}W_t^u, t \in [0,T] \\ X_0^u &= \xi^u, u \in I, \end{aligned} \tag{17}$$

where $\boldsymbol{\xi} = (\xi^u)_u$ an admissible initial condition and $\boldsymbol{\beta} \in L^\infty(I;\mathbb{R}^d)$, $\boldsymbol{\gamma} \in L^\infty(I;\mathbb{R}^d)$, $\boldsymbol{A} \in L^\infty(I,\mathbb{R}^{d\times d})$, $\boldsymbol{B} \in L^\infty(I;\mathbb{R}^{d\times m})$, $G_A \in L^\infty(I\times I;\mathbb{R}^{d\times d})$.

LQ cost functional definition

$$\begin{aligned} J(\boldsymbol{\alpha}) = \int_I \mathbb{E}\Big[\int_0^T & Q^u\big(X_t^u - \int_I \tilde{G}_Q(u,v)\mathbb{E}[X_t^v]\mathrm{d}v\big)\cdot\big(X_t^u - \int_I \tilde{G}_Q(u,v)\mathbb{E}[X_t^v]\mathrm{d}v\big) + \alpha_t^u \cdot R^u \alpha_t^u \\ & + 2\alpha_t^u \cdot \Gamma^u X_t^u + 2\alpha_t^u \cdot \int_I G_I(u,v)\mathbb{E}[X_t^v]\mathrm{d}v\mathrm{d}t \\ & + H^u\big(X_T^u - \int_I \tilde{G}_H(u,v)\mathbb{E}[X_T^v]\mathrm{d}v\big)\cdot\big(X_T^u - \int_I \tilde{G}_H(u,v)\mathbb{E}[X_T^v]\mathrm{d}v\big)\Big]\mathrm{d}u \end{aligned} \tag{18}$$

In the $LQ - NEMFC$, we have the following representation for H and $\hat{\alpha}$.

Hamiltonian H and optimal control α form in the LQ case

$$\begin{aligned} H(u,x,\mu,y,z,a) = &\Big(\beta^u + A^u x + \int_I G_A(u,v)\bar{\mu}^v \mathrm{d}v + B^u a\Big)\cdot y + \gamma^u : z \\ &+ Q^u\big(x - \int_I \tilde{G}_Q(u,v)\bar{\mu}^v dv\big)\cdot\big(x - \int_I \tilde{G}_Q(u,v)\bar{\mu}^v \mathrm{d}v\big) \\ &+ a\cdot R^u a + 2a\cdot\Gamma^u x + 2a\cdot\int_I G_I(u,v)\bar{\mu}^v \mathrm{d}v \end{aligned}$$

such that the unique minimizor is given by :

$$\hat{\alpha}(u,x,\mu,y,z) = -\frac{1}{2}(R^u)^{-1}\Big[(B^u)^\top y + 2\Gamma^u x + 2\int_I G_I(u,v)\bar{\mu}^v \mathrm{d}v\Big]$$

Ansatz form for **Y**

We are looking for an ansatz Y_t^u in the following form :

$$Y_t^u = 2\Big(K^u(t)X_t^u + \int_I \bar{K}_t(u,v)\mathbb{E}[X_t^v]\mathrm{d}v\Big) + \Lambda_t^u\Big), \tag{19}$$

where $K \in C^1\big([0,T]; L^\infty(I;\mathbb{S}_+^d)\big)$, $\bar{K} \in C^1\big([0,T], L^2(I \times I; \mathbb{R}^{d\times d})\big)$ and $\Lambda \in C^1\big([0,T]; L^2(I;\mathbb{R}^d)\big)$ are to be determined through infinite dimensional Ricatti equations.

→ We inject the form (19) in (20) and we end up with a triangular Ricatti system for K, $\bar{K}$ and Λ for which we can prove existence and uniqueness.
→ Finally, we can show existence and uniqueness of the following collection of SDE :

$$\begin{cases} dX_t^u = \Big(\beta^u - B^u(R^u)^{-1}(B^u)^\top\Lambda_t^u + \Big(A^u - B^u(R^u)^{-1}\big((B^u)^\top K_t^u + \Gamma^u\big)\Big)X_t^u \\ \qquad + \int_I \Big(G_A(u,v) - B^u(R^u)^{-1}\big((B^u)^\top \bar{K}_t(u,v) + G_I(u,v)\big)\Big)\mathbb{E}[X_t^v]\mathrm{d}v\Big)\mathrm{d}t + \gamma^u \mathrm{d}W_t^u, \\ X_0^u = \xi^u, \end{cases}$$

Linear Quadratic Graphon MFC

Optimal control α in the L-Q case

$$\alpha_t^u = S^u(t)X_t^u + \int_I \bar{S}^{uv}(t)\mathbb{E}[X_t^v]\mathrm{d}v + \Gamma^u(t), \tag{20}$$

where $\boldsymbol{S} = (S^u)_u$, $\bar{\boldsymbol{S}} = (\bar{S}^{uv})_{u,v}$ and $\boldsymbol{\Gamma} = (\Gamma^u)_u$ are deterministic functions, expressed in terms of K,$\bar{K}$ and Λ given by :

$$\begin{cases} S^u(t) = -(R^u)^{-1}\Big((B^u)^\top K_t^u + \Gamma^u\Big), \\ \bar{S}^{uv}(t) = -(R^u)^{-1}\Big((B^u)^\top \bar{K}_t(u,v) + G_I(u,v)\Big) \\ \Gamma^u(t) = -(R^u)^{-1}(B^u)^\top \Lambda_t^u \end{cases}$$

We consider the following model :

$$dX_t^u = [\kappa(X_t^u - \int_I \tilde{G}_\kappa(u,v)\mathbb{E}[X_t^v]dv) + \alpha_t^u]dt + \sigma^u dW_t^u,$$
$$X_0^u = \xi^u, \tag{21}$$

with $\tilde{G}_\kappa$ a bounded, symmetric measurable function from $I \times I$ into $\mathbb{R}$., $\sigma^u > 0$ and $\alpha = (\alpha^u)$ the control process. The initial condition $\xi = (\xi^u)_u$ is assumed to be admissible. The aim of the central bank is then to minimize over α the following cost functional :

Cost functional $J(\boldsymbol{\alpha})$

$$J(\boldsymbol{\alpha}) = \int_I \mathbb{E}\Big[\int_0^T \Big[\eta^u(X_t^u - \int_I \tilde{G}_\eta(u,v)\mathbb{E}[X_t^v]\mathrm{d}v)^2 + q^u\alpha_t^u(X_t^u - \int_I G_q(u,v)\mathbb{E}[X_t^v]\mathrm{d}v) + |\alpha_t^u|^2\Big]\mathrm{d}t$$
$$+ r^u(X_T^u - \int_I \tilde{G}_r(u,v)\mathbb{E}[X_T^v]\mathrm{d}v)^2\Big]du \tag{22}$$

Optimal control form in systemic risk model

Following (20), we end up with the following optimal control form :

$$\hat{\alpha}_t^u = -(K_t^u + \frac{q^u}{2})(X_t^u - \int_I G_Q(u,v)\mathbb{E}[X_t^v]\mathrm{d}v) - \int_I \Big(\bar{K}_t(u,v) + K_t^u G_Q(u,v)\Big)\mathbb{E}[X_t^v]\mathrm{d}v \tag{23}$$

• Setting the coefficients independant of $u \in I$ and $\tilde{G}_\kappa \equiv \tilde{G}_\eta \equiv \tilde{G}_r \equiv G_Q$, we recover the classical mean-field result see for systemic risk (see [5]) with $\bar{K}_t \equiv -K_t$ and the optimal control is given by :

$$\hat{\alpha}_t = -(K_t + \frac{q}{2})(X_t - \mathbb{E}[X_t])$$

$\rightarrow$ We therefore have the additional term $\int_I \left(\bar{K}_t(u,v) + K_t^u G_Q(u,v)\right) \mathbb{E}[X_t^v]\mathrm{d}v$ which cannot be easily analyzed as we can't solve explicitly $\bar{K}_t^{uv}$.

Conclusion of our work

- We provide a natural extension to the classical MFC problem in the context of non exchangeable interactions by considering the space $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$.

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- We provide a natural extension to the classical MFC problem in the context of non exchangeable interactions by considering the space $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$.
- It leads to the study of a collection indexed by $u \in I$ of fully coupled FBSDE for which we are able to prove existence and uniqueness under standard assumptions on the model coefficients.
- We provide a semi-analytic form of the LQ graphon model as it leads to the study of rectangular Ricatti equations which cannot be solved explicitly evene in simplified models.

On the extended NE-MFC problem

We can extend our work to extended MFC of non exchangeable systems where the dynamics are given by :

$$dX_t^u = b(u, X_t^u, \alpha_t^u, (\mathbb{P}_{(X_t^v,\alpha_t^v)})_{v\in I})\mathrm{d}t + \sigma(u, X_t^u, \alpha_t^u, (\mathbb{P}_{(X_t^v,\alpha_t^v)})_{v\in I})\mathrm{d}W_t^u, \quad 0 \leqslant t \leqslant T, u \in I,$$
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On the heuristic link between $(\mathbf{Y}, \mathbf{Z})$ and the value function v

Denoting by v the value function associated to the optimization problem (3) and (4) , it is shown in [1] that v is law invariant and therefore can be defned on $L^2(\mathcal{P}_2(\mathbb{R}^d))$ and one can heuristically expect that :

$$Y_t^u = \partial \frac{\delta}{\delta m} v\big(t, (\mathbb{P}_{X_t^v})_{v\in I}\big)(u, X_t^u)$$

Future works on $NE - MFC$

- Study of the convergence of the N-agent system towards the limit candidate either by showing convergence of value functions of both problems or convergence of optimal controls.
- Discrete time version of the Maximum principle and the DPP on the *NE-MFC* problem.
- Study of the $NE - MFC$ with common noise.
- Numerical algorithms in the context of a finite number of players :

  (1) In a model-based setting : Learning optimal controls $\boldsymbol{\alpha} = (\alpha^{1,N}, \ldots, \alpha^{N,N})$ and value function V_N through DL algorithms.

  (2) In a model-free setting : Learning optimal controls $\boldsymbol{\alpha} = (\alpha^{1,N}, \ldots, \alpha^{N,N})$ and value function V_N through RL algorithms.

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Thank you for your attention